SCOTTISH QUALIFICATIONS AUTHORITY

2004 Mathematics

Advanced Higher

Finalised Marking Instructions

Solutions to Advanced Higher Mathematics Paper

1. (a)

$$f(x) = \cos^2 x\, e^{\tan x}$$

$$f'(x) = 2(-\sin x)\cos x\, e^{\tan x} + \cos^2 x\ \sec^2 x\, e^{\tan x}$$

1 for Product Rule

2 for accuracy

$$= \left(1 - \sin 2x\right)e^{\tan x}$$

$$f'\left(\frac{\pi}{4}\right) = \left(1 - \sin\frac{\pi}{2}\right)e^{\tan \pi/4} = 0.$$ **1**

(b)

$$g(x) = \frac{\tan^{-1} 2x}{1 + 4x^2}$$

$$g'(x) = \frac{\frac{2}{1+4x^2}(1 + 4x^2) - \tan^{-1} 2x\,(8x)}{(1 + 4x^2)^2}$$

1 for Product Rule

2 for accuracy

$$= \frac{2 - 8x\tan^{-1} 2x}{(1 + 4x^2)^2}$$

2.

$$\left(a^2 - 3\right)^4 = \left(a^2\right)^4 + 4\left(a^2\right)^3(-3) + 6\left(a^2\right)^2(-3)^2 + 4\left(a^2\right)(-3)^3 + (-3)^4$$

1 for binomial coefficients

$$= a^8 - 12a^6 + 54a^4 - 108a^2 + 81$$ **1 for powers**

1 for coefficients

3.

$$x = 5\cos\theta \Rightarrow \frac{dx}{d\theta} = -5\sin\theta$$

$$y = 5\sin\theta \Rightarrow \frac{dy}{d\theta} = 5\cos\theta$$ **1**

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{5\cos\theta}{-5\sin\theta}$$ **1**

When $\theta = \frac{\pi}{4}, \frac{dy}{dx} = -\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = -1,$ **1**

$x = \frac{5}{\sqrt{2}}, y = \frac{5}{\sqrt{2}}$ **1**

so an equation of the tangent is

$$y - \frac{5}{\sqrt{2}} = -\left(x - \frac{5}{\sqrt{2}}\right)$$ **1**

$$\text{i.e. } x + y = 5\sqrt{2}.$$

4.

$$z^2(z+3) = (1+4i-4)(1+2i+3) \quad \textbf{1 for a method}$$
$$= (-3+4i)(4+2i)$$
$$= -20+10i \quad \textbf{1}$$

$$z^3+3z^2-5z+25 = z^2(z+3)-5z+25 \quad \textbf{1 for a method}$$
$$= -20+10i-5-10i+25 = 0 \quad \textbf{1}$$

Note: direct substitution of $1+2i$ *into* $z^3+3z^2-5z+25$ *was acceptable.*

Another root is the conjugate of z, i.e. $1-2i$. **1**
The corresponding quadratic factor is $((z-1)^2+4) = z^2-2z+5$.

$$z^3+3z^2-5z+25 = (z^2-2z+5)(z+5)$$
$$z = -5 \quad \textbf{1}$$

Note: any valid method was acceptable.

5.

$$\frac{1}{x^2-x-6} = \frac{A}{x-3}+\frac{B}{x+2} \quad \textbf{1 for method}$$
$$= \frac{1}{5(x-3)}-\frac{1}{5(x+2)} \quad \textbf{1}$$

$$\int_0^1 \frac{1}{x^2-x-6}\,dx = \frac{1}{5}\int_0^1 \frac{1}{|x-3|}-\frac{1}{|x+2|}\,dx \quad \textbf{1 for method}$$

1 for accuracy

$$= \frac{1}{5}\left[\ln|x-3|-\ln|x+2|\,\right]_0^1 \quad \textbf{1}$$
$$= \frac{1}{5}\left[\ln\frac{|x-3|}{|x+2|}\right]_0^1$$
$$= \frac{1}{5}\left[\ln\frac{2}{3}-\ln\frac{3}{2}\right] \quad \textbf{1}$$
$$= \frac{1}{5}\ln\frac{4}{9} \approx -0.162$$

6.

$$M_1 = \begin{pmatrix}\cos\frac{\pi}{2} & -\sin\frac{\pi}{2}\\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2}\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \quad \textbf{2}$$

$$M_2 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad \textbf{1}$$

$$M_2M_1 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix} \quad \textbf{1}$$

The transformation represented by M_2M_1 is reflection in $y = -x$. **1**

7.

$$f(x) = e^x \sin x \qquad f(0) = 0$$

$$f'(x) = e^x \sin x + e^x \cos x \qquad f'(0) = 1 \qquad \textbf{1}$$

$$f''(x) = e^x \sin x + e^x \cos x - e^x \sin x + e^x \cos x \qquad f''(0) = 2 \qquad \textbf{1}$$

$$= 2e^x \cos x$$

$$f'''(x) = 2e^x \cos x - 2e^x \sin x \qquad f'''(0) = 2 \qquad \textbf{1}$$

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \ldots \qquad \textbf{1}$$

$$e^x \sin x = x + x^2 + \tfrac{1}{3}x^3 - \ldots \qquad \textbf{1}$$

OR

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \qquad \textbf{1}$$

$$\sin x = x - \frac{x^3}{3!} + \ldots \qquad \textbf{1}$$

$$e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right) \textbf{1 – method}$$

$$= x - \frac{x^3}{6} + x^2 - \frac{x^4}{6} + \frac{x^3}{2} - \frac{x^5}{12} + \frac{x^4}{6} + \ldots \qquad \textbf{1}$$

$$= x + x^2 + \frac{x^3}{3} - \ldots \qquad \textbf{1}$$

8.

$$231 = 13 \times 17 + 10 \qquad \textbf{1 for method}$$

$$17 = 1 \times 10 + 7$$

$$10 = 1 \times 7 + 3$$

$$7 = 2 \times 3 + 1 \qquad \textbf{1}$$

Thus the highest common factor is 1.

$$1 = 7 - 2 \times 3$$

$$= 7 - 2 \times (10 - 7) = 3 \times 7 - 2 \times 10 \qquad \textbf{1 for method}$$

$$= 3 \times (17 - 10) - 2 \times 10 = 3 \times 17 - 5 \times 10$$

$$= 3 \times 17 - 5 \times (231 - 13 \times 17) = 68 \times 17 - 5 \times 231. \qquad \textbf{1}$$

So $x = -5$ and $y = 68$.

9.

$$x = (u - 1)^2 \Rightarrow dx = 2(u - 1)\,du \qquad \textbf{1}$$

$$\int \frac{1}{(1 + \sqrt{x})^3}\,dx = \int \frac{2(u - 1)}{u^3}\,du \qquad \textbf{1}$$

$$= 2\int (u^{-2} - u^{-3})\,du \qquad \textbf{1}$$

$$= 2\left(\frac{-1}{u} + \frac{1}{2u^2}\right) + c \qquad \textbf{1}$$

$$= \left(\frac{1}{(1 + \sqrt{x})^2} - \frac{2}{(1 + \sqrt{x})}\right) + c \qquad \textbf{1}$$

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10. $f(x) = x^4 \sin 2x$ so

$$f(-x) = (-x)^4 \sin(-2x) \qquad \mathbf{1}$$
$$= -x^4 \sin 2x \qquad \mathbf{1}$$
$$= -f(x)$$

So $f(x) = x^4 \sin 2x$ is an odd function. **1**

Note: a sketch given with a comment and correct answer, give full marks.
A sketch without a comment, gets a maximum of two marks.

11.

$$V = \int_a^b \pi y^2\, dx \qquad \mathbf{1}$$
$$= \pi \int_0^1 e^{-4x} dx$$

1 for applying formula
1 for accuracy

$$= \pi \left[-\frac{e^{-4x}}{4} \right]_0^1 \qquad \mathbf{1}$$
$$= \pi \left[\frac{-1}{4e^4} + \frac{1}{4} \right] \qquad \mathbf{1}$$
$$= \frac{\pi}{4} \left[1 - \frac{1}{e^4} \right] \approx 0.7706$$

12.

$$\text{LHS} = \frac{d}{dx}(xe^x) = xe^x + 1e^x = (x+1)e^x$$
$$\text{RHS} = (x+1)e^x \qquad \mathbf{1}$$

So true when $n = 1$.

Assume $\frac{d^k}{dx^k}(xe^x) = (x+k)e^x$ **1**

Consider
$$\frac{d^{k+1}}{dx^{k+1}}(xe^x) = \frac{d}{dx}\left(\frac{d^k}{dx^k}(xe^x)\right)$$
$$= \frac{d}{dx}\left((x+k)e^x\right) \qquad \mathbf{1}$$
$$= e^x + (x+k)e^x \qquad \mathbf{1}$$
$$= \left(x + (k+1)\right)e^x$$

So true for k means it is true for $(k+1)$, therefore it is true for all integers $n \geqslant 1$. **1**

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13. (a)

$$y = \frac{x-3}{x+2} = 1 - \frac{5}{x+2}$$ **1**

Vertical asymptote is $x = -2$. **1**

Horizontal asymptote is $y = 1$. **1**

(b)

$$\frac{dy}{dx} = \frac{5}{(x+2)^2}$$ **1**

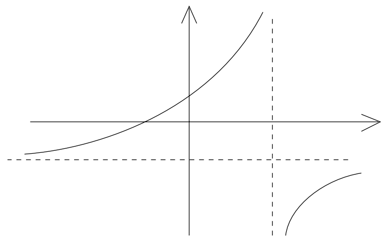
$$\neq 0$$ **1**

(c)

$$\frac{d^2y}{dx^2} = \frac{-10}{(x+2)^3} \neq 0$$ **1**

So there are no points of inflexion. **1**

(d)



1

The asymptotes are $x = 1$ and $y = -2$. **1**

The domain must exclude $x = 1$. **1**

Note: candidates are not required to obtain a formula for f^{-1}.

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14. (a)

$$\overrightarrow{AB} = -\mathbf{i} + 2\mathbf{j} - 4\mathbf{k},\ \overrightarrow{AC} = 0\mathbf{i} + \mathbf{j} - 3\mathbf{k}$$ **1**

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -4 \\ 0 & 1 & -3 \end{vmatrix} = -2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$$ **1 for method** / **1 for accuracy**

$$-2x - 3y - z = c\,(= -2 + 0 - 3 = -5)$$

i.e. an equation for π_1 is $2x + 3y + z = 5$. **1**

Let an angle be θ, then

$$\cos\theta = \frac{(2\mathbf{i} + 3\mathbf{j} + \mathbf{k}).(\mathbf{i} + \mathbf{j} - \mathbf{k})}{\sqrt{4+9+1}\sqrt{1+1+1}}$$ **1**

$$= \frac{2+3-1}{\sqrt{14 \times 3}}$$ **1**

$$= \frac{4}{\sqrt{42}}$$

$$\theta \approx 51{\cdot}9°$$ **1**

Note: an acute angle is required.

(b) Let $\frac{x - 11}{4} = \frac{y - 15}{5} = \frac{z - 12}{2} = t$.

Then $x = 4t + 11; y = 5t + 15; z = 2t + 12$ **1**

$$(4t + 11) + (5t + 15) - (2t + 12) = 0$$

$$7t = -14 \Rightarrow t = -2$$ **1**

$$x = 3; y = 5 \text{ and } z = 8.$$ **1**

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15. (a)

$$x\frac{dy}{dx} - 3y = x^4$$

$$\frac{dy}{dx} - \frac{3}{x}y = x^3$$ **1**

Integrating factor is $e^{\int -\frac{3}{x}dx}$ **1**

$$= x^{-3}.$$ **1**

$$\frac{1}{x^3}\frac{dy}{dx} - \frac{3}{x^4}y = 1$$

$$\frac{d}{dx}\left(\frac{1}{x^3}y\right) = 1$$ **1**

$$\frac{y}{x^3} = x + c$$ **1**

$$y = (x + c)x^3$$

$y = 2$ when $x = 1$, so

$$2 = 1 + c$$ **1**

$$c = 1$$

$$y = (x + 1)x^3$$ **1**

(b)

$$y\frac{dy}{dx} - 3x = x^4$$

$$y\frac{dy}{dx} = x^4 + 3x$$ **1**

$$\int y\,dy = \int (x^4 + 3x)\,dx$$ **1**

$$\frac{y^2}{2} = \frac{x^5}{5} + \frac{3x^2}{2} + c'$$ **1**

When $x = 1, y = 2$ so $c' = 2 - \frac{1}{5} - \frac{3}{2} = \frac{3}{10}$ and so

$$y = \sqrt{2\left(\frac{x^5}{5} + \frac{3x^2}{2} + \frac{3}{10}\right)}.$$ **1**

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16. (a) The series is arithmetic with $a = 8, d = 3$ and $n = 17$. **1**

$$S = \frac{n}{2}\{2a + (n-1)d\} = \frac{17}{2}\{16 + 16 \times 3\} = 17 \times 32 = 544$$ **1**

(b) $a = 2, S_3 = a + ar + ar^2 = 266$. Since $a = 2$ **1**

$$r^2 + r + 1 = 133$$ **1**

$$r^2 + r - 132 = 0$$

$$(r-11)(r+12) = 0$$

$$r = 11 \text{ (since terms are positive)}.$$ **1**

Note: other valid equations could be used.

(c)

$$2(2a + 3 \times 2) = a\left(1 + 2 + 2^2 + 2^3\right)$$ **1,1**

$$4a + 12 = 15a$$

$$11a = 12$$

$$a = \frac{12}{11}$$ **1**

The sum $S_B = \frac{12}{11}\left(2^n - 1\right)$ and $S_A = \frac{n}{2}\left(\frac{24}{11} + 2(n-1)\right) = n\left(\frac{1}{11} + n\right)$.

1 for a valid strategy

n	4	5	6	7
S_B	$\frac{180}{11}$	$\frac{372}{11}$	$\frac{756}{11}$	$\frac{1524}{11}$
S_A	$\frac{180}{11}$	$\frac{280}{11}$	$\frac{402}{11}$	$\frac{546}{11}$

The smallest n is 7. **1**