SCOTTISH
QUALIFICATIONS
AUTHORITY

2006 Mathematics

Advanced Higher

Finalised Marking Instructions

General Marking Principles

These principles describe the approach taken when marking Advanced Higher Mathematics papers. For more detailed guidance please refer to the detailed Marking Instructions.

1 The main principle is to give credit for the skills demonstrated and the criteria met. Failure to have a correct method may not preclude a candidate gaining credit for their solution.

2 The answer to one part of a question, even if incorrect, is accepted as a basis for subsequent dependent parts of the question.

3 The following are not penalised:

- working subsequent to a correct answer (unless it provides firm evidence that the requirements of the question have not been met)
- legitimate variation in numerical values / algebraic expressions.

4 Full credit will only be given where the solution contains appropriate working. Where the correct answer might be obtained by inspection or mentally, credit may be given.

5 Sometimes the method to be used in a particular question is explicitly stated; no credit will be given where a candidate obtains the correct answer by an alternative method.

6 Where the method to be used in a particular question is not explicitly stated, full credit will be given for an alternative valid method.

In the detailed Marking Instructions which follow, marks are shown alongside the line for which they are awarded. There are two codes used, M and E. M indicates a method mark, so in question 1, 1M, 1, 1 means a method mark for the product rule (and then a mark for each of the terms). E is shorthand for error. In question 3, 2E1, means that a correct answer is awarded 2 marks but that 1 mark is deducted for each error.

Advanced Higher Mathematics 2006
Marking Instructions

1.

$$\text{Let } A = \begin{pmatrix} 2 & x \\ -1 & 3 \end{pmatrix}.$$

$$\det A = 6 + x \qquad \mathbf{1}$$

$$A^{-1} = \frac{1}{6 + x}\begin{pmatrix} 3 & -x \\ 1 & 2 \end{pmatrix} \qquad \mathbf{2E1}$$

The matrix is singular when $x = -6$. **1**

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2. (a)

$$f(x) = 2\tan^{-1}\sqrt{1 + x}$$

$$f'(x) = \frac{2\frac{d}{dx}\left((1 + x)^{1/2}\right)}{1 + (1 + x)} \qquad \mathbf{1,1}$$

$$= \frac{1}{(2 + x)\sqrt{1 + x}} \qquad \mathbf{1}$$

(b)

$$y = \frac{1 + \ln x}{3x}$$

$$\frac{dy}{dx} = \frac{\frac{1}{x}3x - (1 + \ln x)\,3}{9x^2} \qquad \mathbf{M1,1}$$

$$= \frac{-\ln x}{3x^2} \qquad \mathbf{1}$$

3.

$$z = -i + \frac{1}{1 - i} = -i + \frac{1}{1 - i} \times \frac{1 + i}{1 + i} \qquad \mathbf{1}$$

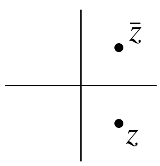
$$= -i + \frac{1 + i}{2} \qquad \mathbf{1}$$

$$= \frac{1}{2} - \frac{1}{2}i \qquad \mathbf{1}$$

i.e. $x = \frac{1}{2}$ and $y = -\frac{1}{2}$.

$$|z| = \sqrt{\tfrac{1}{2}^2 + (-\tfrac{1}{2})^2} = \tfrac{1}{2}\sqrt{2} \qquad \mathbf{1}$$

$$\arg z = \tan^{-1}\left(\frac{-\frac{1}{2}}{\frac{1}{2}}\right) = \frac{-\pi}{4}\left(\text{or } \frac{7\pi}{4}\right) \qquad \mathbf{1}$$



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4.

$$xy - x = 4$$

$$\frac{d}{dx}(xy) - 1 = 0 \qquad \textbf{M1}$$

$$x\frac{dy}{dx} + y - 1 = 0$$

$$\frac{dy}{dx} = \frac{1 - y}{x} \qquad \textbf{1}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1 - y}{x}\right)$$

$$= \frac{-x\frac{dy}{dx} - (1 - y)}{x^2} \qquad \textbf{1}$$

$$= \frac{-x\left(\frac{1 - y}{x}\right) - (1 - y)}{x^2} \qquad \textbf{1}$$

$$= \frac{2(y - 1)}{x^2} \qquad \textbf{1}$$

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5. Let the fixed point be λ, then

$$\lambda = \frac{1}{2}\left(\lambda + \frac{2}{\lambda^2}\right) \qquad \textbf{1}$$

$$\frac{\lambda}{2} = \frac{1}{\lambda^2} \qquad \textbf{1}$$

$$\lambda^3 = 2$$

$$\lambda = \sqrt[3]{2} \qquad \textbf{1}$$

6.

$$\frac{d}{dx}\left(x^4 - x^2 + 1\right) = 4x^3 - 2x \qquad \textbf{1}$$

Thus

$$\int \frac{12x^3 - 6x}{x^4 - x^2 + 1}\,dx = 3\int \frac{4x^3 - 2x}{x^4 - x^2 + 1}\,dx \qquad \textbf{1}$$

$$= 3\ln\left(x^4 - x^2 + 1\right) + c \qquad \textbf{1}$$

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7. (a) $n^3 - n = n(n^2 - 1) = (n - 1)n(n + 1)$ **1**

Since $n^3 - n$ is the product of 3 consecutive integers, it is divisible by 3 and also by 2 so it is divisible by 6. **2**

(b) Taking $n = 2, n^3 + n + 5 = 15.$ **1**

[Or, $n = 5, n^3 + n + 5 = 125 + 5 + 5$ and is divisible by 5.]

The statement is false. **1**

8.

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$$

$$\text{A.E.} \qquad m^2 + 2m + 2 = 0 \qquad \mathbf{1}$$

$$m = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i \qquad \mathbf{1}$$

General solution is

$$y = e^{-x}(A\cos x + B\sin x) \qquad \mathbf{1}$$

$$y = 0 \text{ when } x = 0 \Rightarrow 0 = A \qquad \mathbf{1}$$

$$\frac{dy}{dx} = -e^{-x}B\sin x + e^{-x}B\cos x \qquad \mathbf{1}$$

$$2 = 0 + B \qquad \mathbf{1}$$

The solution is $y = 2e^{-x}\sin x$.

9.

$$\left.\begin{array}{rrr} 2 & -1 & 2 \\ 1 & 1 & -2 \\ 1 & -2 & 4 \end{array}\right| \begin{array}{r} 1 \\ 2 \\ -1 \end{array} \Rightarrow \left.\begin{array}{rrr} 2 & -1 & 2 \\ 0 & -3 & 6 \\ 0 & 3 & -6 \end{array}\right| \begin{array}{r} 1 \\ -3 \\ 3 \end{array} \qquad \begin{array}{c} \\ \mathbf{1} \\ \mathbf{1} \end{array}$$

$$\Rightarrow \left.\begin{array}{rrr} 2 & -1 & 2 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{array}\right| \begin{array}{r} 1 \\ -3 \\ 0 \end{array} \qquad \mathbf{1}$$

Thus $z = t$, **1**

$y = 1 + 2t$ and $x = 1$. **1**

10.

$$x = T^3 - 90T^2 + 2400T$$

$$\frac{dx}{dT} = 3T^2 - 180T + 2400 \qquad \mathbf{1}$$

$$= 0 \text{ at stationary values.}$$

$$3(T - 40)(T - 20) = 0$$

$$\text{i.e. } T = 20 \text{ or } T = 40 \qquad \mathbf{1}$$

$$\frac{d^2x}{dT^2} = 6T - 180$$

$\Rightarrow T = 20$ is a local maximum (and $T = 40$ is a local minimum). **1**

$$x(20) = 20000$$

Check end of interval values

$$x(10) = 1000 - 9000 + 24000 = 16000$$

$$x(60) = 216000 - 324000 + 144000 = 36000$$

So the best result is when $T = 60$. **1**

11.

$$1 + \cot^2\theta = \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} = \operatorname{cosec}^2\theta \qquad \mathbf{1}$$

$$y = \cot^{-1} x$$

$$\cot y = x$$

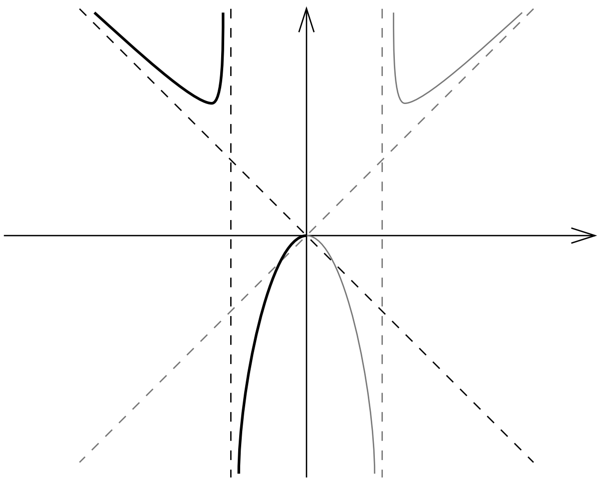
$$-\operatorname{cosec}^2 y \frac{dy}{dx} = 1 \qquad \mathbf{1}$$

$$-(1 + \cot^2 y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{1 + \cot^2 y} \qquad \mathbf{1}$$

$$= -\frac{1}{1 + x^2} \qquad \mathbf{1}$$

12.



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[Asymptotes not required on the diagram.]
The other asymptotes are $y = -x$ and $x = -1$ **1, 1**

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13. For $n = 1$, LHS = AB and RHS = BA. These are equal, so true for $n = 1$. **1**

Assume true for $n = k$, i.e. $A^kB = BA^k$. **1**

Consider $n = k + 1$.

$$\text{LHS} = A^{k+1}B = A^kAB \qquad \mathbf{1}$$

$$= A^kBA \qquad \mathbf{1}$$

$$= BA^kA$$

$$= BA^{k+1} = \text{RHS} \qquad \mathbf{1}$$

Thus if true for $n = k$, true for $n = k + 1$ and hence true for all $n \geqslant 1$.

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14. (a)

$$f(-x) = (-x)^2 \sin(-x) \quad \textbf{1}$$
$$= x^2(-\sin x) = -x^2 \sin x = -f(x) \quad \textbf{1}$$

i.e. $f(x)$ is odd **1**

(b)

$$\int x^2 \sin x\, dx = x^2 \int \sin x\, dx - \int \left(2x \int \sin x\ dx\right) dx \quad \textbf{1}$$
$$= x^2(-\cos x) - \int -2x \cos x\, dx \quad \textbf{1}$$
$$= -x^2 \cos x + \left(2x \int \cos x\, dx - \int 2 \sin x\, dx\right) \quad \textbf{1}$$
$$= -x^2 \cos x + 2x \sin x + 2\cos x + c \quad \textbf{1}$$

(c) Since $f(x)$ is odd and the x limits are symmetrical, the area is given by

$$2\int_0^{\pi/4} f(x)\, dx = 2\left[-x^2 \cos x + 2x \sin x + 2\cos x\right]_0^{\pi/4} \quad \textbf{M1}$$
$$= 2\left\{\left[-\left(\frac{\pi}{4}\right)^2 \cos\frac{\pi}{4} + \frac{2\pi}{4}\sin\frac{\pi}{4} + 2\cos\frac{\pi}{4}\right] - [0+0+2]\right\} \quad \textbf{1}$$
$$= 2\left(\frac{\pi}{2\sqrt{2}} + \sqrt{2} - \frac{\pi^2}{16\sqrt{2}} - 2\right) \quad \textbf{1}$$
$$= \frac{1}{8\sqrt{2}}\left(8\pi - \pi^2 + 32 - 32\sqrt{2}\right)(\approx 0{\cdot}1775)$$

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15. The line $\dfrac{x+1}{2} = \dfrac{y-2}{1} = \dfrac{z}{-1}$ has direction $2\mathbf{i} + \mathbf{j} - \mathbf{k}$. **1**

An equation of the plane is $2x + y - z = k$. **1**

Using (1,1,0), $k = 2 + 1 = 3$.

i.e. $2x + y - z = 3$. **1**

$$\text{Put } \frac{x+1}{2} = \frac{y-2}{1} = \frac{z}{-1} = t.$$

Hence $x = -1 + 2t$; $y = 2 + t$; $z = -t$. **1**

Substitute into $2x + y - z = 3$

$$2(-1+2t) + (2+t) + t = 3 \quad \textbf{1}$$
$$6t = 3 \Rightarrow t = \tfrac{1}{2} \quad \textbf{1}$$

The point of intersection, Q, is $\left(0,\ 2\tfrac{1}{2},\ -\tfrac{1}{2}\right)$ **1**

The shortest distance is PQ.

i.e. $\sqrt{1^2 + \left(\dfrac{3}{2}\right)^2 + \left(\dfrac{-1}{2}\right)^2} = \sqrt{1 + \dfrac{9}{4} + \dfrac{1}{4}} = \sqrt{\dfrac{7}{2}} \approx 1.87.$ **M1,1**

This is the shortest distance because PQ is perpendicular to L. **1**

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16. (a)

$$r = \frac{\frac{x(x+1)^2}{(x-2)^2}}{\frac{x(x+1)}{(x-2)}} = \frac{(x+1)}{(x-2)} \qquad \mathbf{1}$$

$$u_n = ar^{n-1} \qquad \mathbf{1}$$

$$= \frac{x(x+1)^n}{(x-2)^n} \qquad \mathbf{1}$$

(b)

$$S_n = \frac{a(r^n - 1)}{r - 1} \qquad \mathbf{1}$$

$$= \frac{x(x+1)}{(x-2)} \frac{\left(\frac{(x+1)^n}{(x-2)^n} - 1\right)}{\left(\frac{x+1}{x-2} - 1\right)} \qquad \mathbf{2}$$

$$= \frac{1}{3}x(x+1)\left(\frac{(x+1)^n}{(x-2)^n} - 1\right)$$

(c) For a sum to infinity, $-1 < r < 1$, i.e. $r^2 < 1$

$$\frac{(x+1)^2}{(x-2)^2} < 1 \qquad \mathbf{1}$$

$$x^2 + 2x + 1 < x^2 - 4x + 4 \qquad \mathbf{1}$$

$$6x < 3$$

$$\text{i.e.} \quad x < \frac{1}{2} \qquad \mathbf{1}$$

$$S = \frac{a}{1 - r} = \frac{x(x+1)}{(x-2)\left(1 - \frac{x+1}{x-2}\right)} \qquad \mathbf{1}$$

$$= \frac{-x(x+1)}{3}$$

17. (a)

$$\int \cos^2 x \sin^2 x \, dx = \int \cos^2 x (1 - \cos^2 x) \, dx \qquad \mathbf{1}$$

$$= \int \cos^2 x \, dx - \int \cos^4 x \, dx$$

(b)

$$\int_0^{\pi/4} \cos^4 x \, dx = \int_0^{\pi/4} \cos x \cos^3 x \, dx$$

$$= \left[\cos^3 x \int \cos x \, dx\right]_0^{\pi/4} - \int_0^{\pi/4} \left[3\cos^2 x(-\sin x)\sin x\right] dx \qquad \mathbf{1}$$

$$= \left[\cos^3 x \sin x\right]_0^{\pi/4} + 3\int_0^{\pi/4} \cos^2 x \sin^2 x \, dx \qquad \mathbf{1}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^3 \frac{1}{\sqrt{2}} + 3\int_0^{\pi/4} \cos^2 x \sin^2 x \, dx \qquad \mathbf{1}$$

$$= \frac{1}{4} + 3\int_0^{\pi/4} \cos^2 x \sin^2 x \, dx$$

(c) From $\cos 2x = 2\cos^2 x - 1$, we get $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ **1**

$$\int_0^{\pi/4} \cos^2 x \, dx = \int_0^{\pi/4} \frac{1}{2}(1 + \cos 2x) \, dx$$

$$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/4} \qquad \mathbf{1}$$

$$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}\right] - \frac{1}{2}[0] \qquad \mathbf{1}$$

$$= \frac{\pi + 2}{8}$$

(d)

$$\int_0^{\pi/4} \cos^4 x \, dx = \frac{1}{4} + 3\int_0^{\pi/4} \cos^2 x \sin^2 x \, dx$$

$$= \frac{1}{4} + 3\int_0^{\pi/4} \cos^2 x \, dx - 3\int_0^{\pi/4} \cos^4 x \, dx \qquad \mathbf{1}$$

$$4\int_0^{\pi/4} \cos^4 x \, dx = \frac{1}{4} + 3\left(\frac{\pi + 2}{8}\right) = \frac{3\pi + 8}{8} \qquad \mathbf{2E1}$$

$$\int_0^{\pi/4} \cos^4 x \, dx = \frac{3\pi + 8}{32}$$

[END OF MARKING INSTRUCTIONS]