2007 Mathematics

Advanced Higher

Finalised Marking Instructions

General Marking Principles

These principles describe the approach taken when marking Advanced Higher Mathematics papers. For more detailed guidance please refer to the detailed Marking Instructions.

1 The main principle is to give credit for the skills demonstrated and the criteria met. Failure to have a correct method may not preclude a candidate gaining credit for their solution.

2 The answer to one part of a question, even if incorrect, can be accepted as a basis for subsequent dependent parts of the question.

3 The following are not penalised:

- working subsequent to a correct answer (unless it provides firm evidence that the requirements of the question have not been met)
- legitimate variation in numerical values / algebraic expressions.

4 Full credit will only be given where the solution contains appropriate working. Where the correct answer might be obtained by inspection or mentally, credit may be given.

5 Sometimes the method to be used in a particular question is explicitly stated; no credit will be given where a candidate obtains the correct answer by an alternative method.

6 Where the method to be used in a particular question is not explicitly stated in the question paper, full credit is available for an alternative valid method. (Some likely alternatives are included but these should not be assumed to be the only acceptable ones.)

In the detailed Marking Instructions which follow, marks are shown alongside the line for which they are awarded. There are two codes used, M and E. M indicates a method mark, so in question 2(a), M1 means a method mark for the chain rule. E is shorthand for error. For example, 2E1, means that a correct answer is awarded 2 marks but that 1 mark is deducted for each error.

Advanced Higher Mathematics 2007

1.

$$\left(x - \frac{2}{x}\right)^4 = x^4 + 4x^3\left(-\frac{2}{x}\right) + 6x^2\left(-\frac{2}{x}\right)^2 + 4x\left(-\frac{2}{x}\right)^3 + \left(-\frac{2}{x}\right)^4$$ **1 for powers**
1 for coeffs

$$= x^4 - 8x^2 + 24 - \frac{32}{x^2} + \frac{16}{x^4}$$ **2E1**

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2. (a)

$$f(x) = \exp(\sin 2x)$$

$$f'(x) = 2\cos 2x \exp(\sin 2x)$$ **M1,2E1**

(b)

$$y = 4^{(x^2+1)}$$

$$\ln y = \ln\left(4^{(x^2+1)}\right) = \left(x^2 + 1\right)\ln 4$$ **M1**

$$\frac{1}{y}\frac{dy}{dx} = 2x\ln 4$$ **1**

$$\frac{dy}{dx} = 2x\ln 4 \,.\, 4^{(x^2+1)}$$ **1**

Alternative:

$$y = 4^{(x^2+1)}$$

$$4 = e^{\ln 4}$$ **1**

$$y = e^{\ln 4(x^2+1)}$$

$$\frac{dy}{dx} = \ln 4\, 2x\, e^{\ln 4(x^2+1)}$$ **1,1**

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3. $(3 + 3i)^3 = 27 + 81i + 81i^2 + 27i^3 = -54 + 54i$. Thus

$$(3 + 3i)^3 - 18(3 + 3i) + 108 =$$

$$-54 + 54i - 54 - 54i + 108 = 0$$ **1**

Since $3 + 3i$ is a root, $3 - 3i$ is a root. **1**

These give a factor $\left(z - (3 + 3i)\right)\left(z - (3 - 3i)\right) = (z - 3)^2 + 9 = z^2 - 6z + 18$. **1**

$$z^3 - 18z + 108 = \left(z^2 - 6z + 18\right)\left(z + 6\right)$$

The remaining roots are $3 - 3i$ and -6. **1**

4.

$$\frac{2x^2 - 9x - 6}{x(x^2 - x - 6)} = \frac{2x^2 - 9x - 6}{x(x+2)(x-3)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-3} \qquad \mathbf{1}$$

$$2x^2 - 9x - 6 = A(x+2)(x-3) + Bx(x-3) + Cx(x+2)$$

$$\begin{aligned} x = 0 &\Rightarrow -6A = -6 \Rightarrow A = 1 \\ x = -2 &\Rightarrow 10B = 20 \Rightarrow B = 2 \\ x = 3 &\Rightarrow 15C = -15 \Rightarrow C = -1 \end{aligned} \qquad \mathbf{2E1}$$

$$\therefore \frac{2x^2 - 9x - 6}{x(x^2 - x - 6)} = \frac{1}{x} + \frac{2}{x+2} - \frac{1}{x-3}$$

$$\begin{aligned} \int_4^6 \frac{2x^2 - 9x - 6}{x(x^2 - x - 6)} &= \int_4^6 \left(\frac{1}{x} + \frac{2}{x+2} - \frac{1}{x-3}\right) dx \\ &= \left[\ln x + 2\ln(x+2) - \ln(x-3)\right]_4^6 \qquad \mathbf{2E1} \\ &= \left[\ln \frac{x(x+2)^2}{(x-3)}\right]_4^6 \\ &= \ln \frac{6 \times 64}{3} - \ln \frac{4 \times 36}{1} \\ &= \ln \frac{2 \times 64}{4 \times 36} = \ln \frac{8}{9} \qquad \mathbf{1} \end{aligned}$$

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5. (a)

$$AB = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x+2 & x-2 & x+3 \\ -4 & 4 & 2 \\ 2 & -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} x & x & x \\ -6 & 6 & -1 \\ 0 & 0 & 8 \end{pmatrix} \qquad \mathbf{2E1}$$

(b)

$$\det A = 1 \times (2 + 1) - 0 - 1 \times 0 = 3 \qquad \mathbf{1}$$

$$\det AB = x(48 - 0) - x(-48 - 0) + x(0 - 0) = 96x \qquad \mathbf{1}$$

Since $\det AB = \det A \det B$

$$\det B = \frac{\det AB}{\det A} = \frac{96x}{3} = 32x \qquad \mathbf{1}$$

6.

$$f(x) = f'(0) + xf'(0) + \tfrac{x^2}{2!}f''(0) + \tfrac{x^3}{3!}f'''(0) + \ldots \qquad \mathbf{1}$$

$$\cos x = 1 - \tfrac{x^2}{2} + \tfrac{x^4}{24} - \ldots \qquad \mathbf{1}$$

$$f(x) = \frac{1}{2}\left(1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{24} - \ldots\right) \qquad \mathbf{1}$$

$$= \tfrac{1}{2} - x^2 + \tfrac{x^4}{3} - \ldots \qquad \mathbf{1}$$

$$f(3x) = \tfrac{1}{2} - (3x)^2 + \tfrac{1}{3}(3x)^4$$

$$= \tfrac{1}{2} - 9x^2 + 27x^4 - \ldots \qquad \mathbf{1}$$

Alternative for third and fourth marks:

$$\begin{aligned} f(x) &= \tfrac{1}{2}\cos 2x & f(0) &= \tfrac{1}{2} \\ f'(x) &= -\sin 2x & f'(0) &= 0 \\ f''(x) &= -2\cos 2x & f''(0) &= -2 \\ f'''(x) &= 4\sin 2x & f'''(0) &= 0 \\ f''''(x) &= 8\cos 2x & f''''(0) &= 8 \end{aligned}$$

1

In general

$$f(x) = f(0) + f'(0)x + f''(0)\tfrac{x^2}{2} + \ldots$$

Hence

$$\begin{aligned} f(x) &= \tfrac{1}{2} + 0 + (-2)\tfrac{x^2}{2} + 0 + 8\tfrac{x^4}{24} + \ldots \\ &= \tfrac{1}{2} - x^2 + \tfrac{x^4}{3} - \ldots \end{aligned}$$

1

7.

$$\begin{aligned} 599 &= 53 \times 11 + 16 \\ 53 &= 16 \times 3 + 5 \\ 16 &= 5 \times 3 + 1 \end{aligned}$$

1

$$\begin{aligned} 1 &= 16 - 5 \times 3 \\ &= 16 - (53 - 16 \times 3) \times 3 \\ &= 16 \times 10 - 53 \times 3 \\ &= (599 - 53 \times 11) \times 10 - 53 \times 3 \\ &= 599 \times 10 - 53 \times 113 \end{aligned}$$

2E1

Hence $599p + 53q = 1$ when $p = 10$ and $q = -113$. **1**

8.

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^{2x}$$

Auxiliary equation: $m^2 + 6m + 9 = 0$ **1**

So $(m+3)^2 = 0$ giving $m = -3$.

Complementary function:

$$y = (A + Bx)e^{-3x}$$ **1**

For the Particular Integral try $y = ke^{2x}$ **1**

$$\Rightarrow \frac{dy}{dx} = 2ke^{2x}; \frac{d^2y}{dx^2} = 4ke^{2x}$$ **1**

$$4ke^{2x} + 12ke^{2x} + 9ke^{2x} = e^{2x} \Rightarrow 25k = 1$$ **1**

Hence the General Solution is:

$$y = (A + Bx)e^{-3x} + \tfrac{1}{25}e^{2x}$$ **1**

9.

$$\sum_{r=1}^{n}(4-6r) = 4\sum_{r=1}^{n} - 6\sum_{r=1}^{n} r \qquad \textbf{1M}$$

$$= 4n - 3n(n+1) \qquad \textbf{1}$$

$$= n - 3n^2$$

$$\sum_{r=1}^{2q}(4-6r) = 2q - 12q^2 \qquad \textbf{1}$$

$$\sum_{r=q+1}^{2q}(4-6r) = \sum_{r=1}^{2q}(4-6r) - \sum_{r=1}^{q}(4-6r) \qquad \textbf{1M}$$

$$= \left(2q - 12q^2\right) - \left(q - 3q^2\right) \qquad \textbf{1}$$

$$= q - 9q^2.$$

Arithmetic Series could be used, so, for the first two marks:

$$a = -2,\ d = -6 \Rightarrow S_n = \tfrac{n}{2}\left\{2(-2) + (n-1)(-6)\right\} \qquad \textbf{1}$$

$$= -2n - 3n^2 + 3n = n - 3n^2 \qquad \textbf{1}$$

10.

$$1 + x^2 = u \Rightarrow 2x\,dx = du \qquad \textbf{1}$$

$$x = 0 \Rightarrow u = 1; \qquad x = 1 \Rightarrow u = 2 \qquad \textbf{1}$$

$$\int_0^1 \frac{x^3}{(1+x^2)^4}\,dx = \int_1^2 \frac{(u-1)}{2u^4}\,du \qquad \textbf{1}$$

$$= \frac{1}{2}\int_1^2 \left(u^{-3} - u^{-4}\right)du$$

$$= \frac{1}{2}\left[-\frac{1}{2}u^{-2} + \frac{1}{3}u^{-3}\right]_1^2 \qquad \textbf{1}$$

$$= \frac{1}{2}\left[-\frac{1}{8} + \frac{1}{24}\right] - \frac{1}{2}\left[-\frac{1}{2} + \frac{1}{3}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{12} + \frac{1}{6}\right] = \frac{1}{24} \qquad \textbf{1}$$

The volume of revolution is given by $V = \int_a^b \pi y^2 dx$. So in this case

$$V = \pi\int_0^1 \frac{x^3}{(1+x)^4}\,dx = \frac{\pi}{24}. \qquad \textbf{1}$$

Integration by parts could be used for marks three, four and five.

$$\int_1^2 \frac{u-1}{2u^4}\,du = \frac{1}{2}\left[(u-1)\int u^{-4}du - \int 1.\frac{u^{-3}}{-3}du\right]_1^2 \qquad \textbf{1}$$

$$= \frac{1}{2}\left[\frac{u-1}{-3u^3} + \frac{u^{-2}}{(-6)}\right]_1^2 \qquad \textbf{1}$$

$$= \frac{1}{2}\left[\frac{1}{-24} - \frac{1}{24}\right] - \frac{1}{2}\left[0 - \frac{1}{6}\right]$$

$$= -\frac{1}{24} + \frac{1}{12} = \frac{1}{24} \qquad \textbf{1}$$

11.

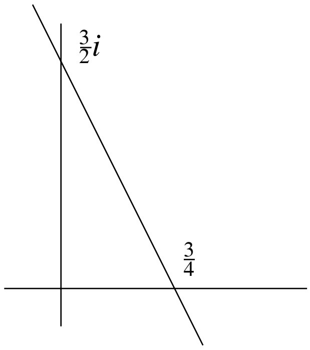
$$|z-2| = |z+i|$$

$$|(x-2)+iy| = |x+(y+1)i| \qquad \mathbf{1}$$

$$(x-2)^2+y^2 = x^2+(y+1)^2 \qquad \mathbf{1}$$

$$-4x+4 = 2y+1$$

$$4x+2y-3 = 0 \qquad \mathbf{1}$$



1

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12. Consider $n = 1$, LHS $= (1+a)$, RHS $= 1+a$ so true for $n = 1$. **1**
Assume that $(1+a)^k \geqslant 1+ka$ and consider $(1+a)^{k+1}$. **1**

$$(1+a)^{k+1} = (1+a)(1+a)^k \qquad \mathbf{1}$$

$$\geqslant (1+a)\left(1+ka\right) \qquad \mathbf{1}$$

$$= 1+a+ka+ka^2$$

$$= 1+(k+1)a+ka^2$$

$$> 1+(k+1)a \text{ since } ka^2 > 0 \qquad \mathbf{1}$$

as required. So since true for $n = 1$, by mathematical induction statement is true for all $n \geqslant 1$.

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13. (a)

$$x = \cos 2t \Rightarrow \frac{dx}{dt} = -2\sin 2t;\; y = \sin 2t \Rightarrow \frac{dy}{dt} = 2\cos 2t \qquad \mathbf{1{,}1}$$

$$\frac{dy}{dx} = \frac{2\cos 2t}{-2\sin 2t} = -\cot 2t \qquad \mathbf{1}$$

When $t = \frac{\pi}{8}$, $x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$; $y = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$; $\frac{dy}{dx} = -1$. **1**

Equation is: $y - \dfrac{1}{\sqrt{2}} = -\left(x - \dfrac{1}{\sqrt{2}}\right)$ i.e. $x+y = \sqrt{2}$. **1**

(b)

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \qquad \mathbf{1M}$$

$$= \frac{2\,\mathrm{cosec}^2 2t}{-2\sin 2t} \qquad \mathbf{2E1}$$

$$= \frac{-1}{\sin^3 2t}$$

$$\sin 2t\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{-\sin 2t}{\sin^3 2t} + \left(\frac{-\cos 2t}{\sin 2t}\right)^2 \qquad \mathbf{1}$$

$$= \frac{-1+\cos^2 2t}{\sin^2 2t} = -1 \qquad \mathbf{1}$$

Alternative for last three marks of (b)

$$\sin 2t\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -\operatorname{cosec}^2 2t + \left(\frac{-\cos 2t}{\sin 2t}\right)^2 \qquad \mathbf{1}$$

$$= \frac{-1}{\sin^2 2t} + \frac{\cos^2 2t}{\sin^2 2t} \qquad \mathbf{1}$$

$$= \frac{-1 + \cos^2 2t}{\sin^2 2t} = -1 \qquad \mathbf{1}$$

14. (a)

$$\frac{dG}{dt} = \frac{25k - G}{25}$$

$$\int \frac{dG}{25k - G} = \int \frac{1}{25}dt \qquad \mathbf{1}$$

$$-\ln(25k - G) = \frac{t}{25} + C \qquad \mathbf{1}$$

When $t = 0, G = 0$, so $C = -\ln 25k$ **1**

$$25k - G = 25ke^{-t/25}$$

$$G = 25k\left(1 - e^{-t/25}\right) \qquad \mathbf{1}$$

(b) When $t = 5, G = 0.6$. Therefore

$$0.6 = 25k\left(1 - e^{-0.2}\right) \qquad \mathbf{1}$$

$$k = 0.6/\left(25\left(1 - e^{-0.2}\right)\right) \approx 0{\cdot}132 \qquad \mathbf{1}$$

(c) When $t = 10$

$$G \approx 3.3\left(1 - e^{-0.4}\right) \qquad \mathbf{1}$$

$$\approx 1.09$$

The claim seems to be justified, **1**

(d) As $t \to \infty, G \to 25k \approx 3.3$ metres **1**

so the limit is 3.6 metres. **1**

Alternative using an Integrating Factor:

(a)

$$\frac{dG}{dt} = \frac{25k - G}{25}$$

$$\frac{dG}{dt} + \frac{G}{25} = k \qquad \mathbf{1}$$

$$\text{IF} = e^{\int \frac{1}{25}dt} = e^{t/25}$$

$$\frac{d}{dt}\left(e^{t/25}G\right) = ke^{t/25}$$

$$e^{t/25}G = k\int e^{t/25}dt$$

$$= k\left(25e^{t/25}\right) + C'$$

$$G = 25k + C'e^{-t/25} \qquad \mathbf{1}$$

When $t = 0, G = 0$, so $C' = -25k$ **1**

$$G = 25k\left(1 - e^{-t/25}\right) \qquad \mathbf{1}$$

15. (a) Equating the x-coordinates: $2 + s = -1 - 2t \Rightarrow s + 2t = -3$ (1)

Equating the y-coordinates: $-s = t \Rightarrow s = -t$ **1**

Substituting in (1): $-t + 2t = -3 \Rightarrow t = -3 \Rightarrow s = 3$. **1**

Putting $s = 3$ in L_1 gives $(5, -3, -1)$ and $t = -3$ in L_2, $(5, -3, -7)$.

As the z coordinates differ, L_1 and L_2 do not intersect. **1**

(b) Directions of L_1 and L_2 are: $\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$. The vector product of these gives the direction of L_3.

$$(\mathbf{i}-\mathbf{j}-\mathbf{k}) \times (-2\mathbf{i}+\mathbf{j}+3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ -2 & 1 & 3 \end{vmatrix} = -2\mathbf{i}-\mathbf{j}-\mathbf{k}$$ **1M,1**

Equation of L_3:

$$\begin{aligned} \mathbf{r} &= \mathbf{i} + \mathbf{j} + 3\mathbf{k} + (-2\mathbf{i} - \mathbf{j} - \mathbf{k})u \\ &= (1 - 2u)\mathbf{i} + (1 - u)\mathbf{j} + (3 - u)\mathbf{k} \end{aligned}$$

Hence L_3 is given by $x = 1 - 2u$, $y = 1 - u$, $z = 3 - u$. **1**

(c) Solving the x and y coordinates of L_3 and L_2:

$$-1 - 2t = 1 - 2u \text{ and } t = 1 - u$$

$$\Rightarrow -1 = 3 - 4u \Rightarrow u = 1 \text{ and } t = 0$$ **1**

The point of intersection, Q, is $(-1, 0, 2)$ since $2 + 3t = 2$ and $3 - u = 2$. **1**

L_1 is $x = 2 + s, y = -s, z = 2 - s$. When $x = 1$, $s = -1$ and hence $y = 1$ and $z = 3$, i.e. P lies on L_1. **1**

(d) $PQ = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. **1**

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16. (a) $\tan^{-1} 2x$ has horizontal asymptotes at $y = \pm\frac{\pi}{2}$. **1,1**

(b)

$$\text{Area} = \int_0^{1/2} \tan^{-1} 2x \, dx$$ **1**

$$= \int_0^{1/2} (\tan^{-1} 2x) \times 1 \, dx$$ **1**

$$= \left[\tan^{-1} 2x \int 1.dx - \int \frac{2}{1 + 4x^2} . x \, dx\right]_0^{1/2}$$

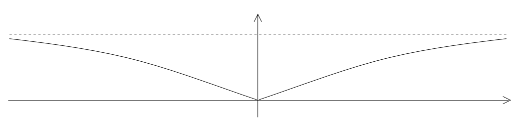
$$= \left[x \tan^{-1} 2x - \frac{1}{4}\int \frac{8x}{1 + 4x^2} dx\right]_0^{1/2}$$

$$= \left[x \tan^{-1} 2x - \frac{1}{4} \ln(1 + 4x^2)\right]_0^{1/2}$$ **2E1**

$$= \left[\frac{1}{2} \tan^{-1} 1 - \frac{1}{4} \ln 2\right] - [0 - 0]$$

$$= \frac{\pi}{8} - \frac{1}{4} \ln 2$$ **1**

(c)



2E1

$$\int_{-1/2}^{1/2} |f(x)| \, dx = 2\int_0^{1/2} \tan^{-1} 2x \, dx$$

$$= \frac{\pi}{4} - \frac{1}{2} \ln 2$$ **1**

[END OF MARKING INSTRUCTIONS]