2009 Mathematics

Advanced Higher

Finalised Marking Instructions

Solutions for AH maths 2009

1. (a)

$$f(x) = (x+1)(x-2)^3$$

$$f'(x) = (x-2)^3 + 3(x+1)(x-2)^2 \qquad \mathbf{1}$$

$$= (x-2)^2\left((x-2) + 3(x+1)\right)$$

$$= (x-2)^2(4x+1) \qquad \mathbf{1}$$

$$= 0 \text{ when } x = 2 \text{ and when } x = -\frac{1}{4}. \qquad \mathbf{1}$$

(b) *Method 1*

$$\frac{x^2}{y} + x = y - 5 \Rightarrow x^2 + xy = y^2 - 5y \qquad \mathbf{1}$$

$$2x + x\frac{dy}{dx} + y = 2y\frac{dy}{dx} - 5\frac{dy}{dx} \qquad \mathbf{2E1}$$

$$6 + 3\frac{dy}{dx} - 1 = -2\frac{dy}{dx} - 5\frac{dy}{dx}$$

$$5 = -10\frac{dy}{dx} \Rightarrow \boldsymbol{\frac{dy}{dx} = \frac{-1}{2}} \qquad \mathbf{1}$$

Note: a candidate may obtain $\frac{dy}{dx} = \frac{2x+y}{2y-x-5}$ and then substitute.

Method 2

$$\frac{x^2}{y} + x = y - 5$$

$$\frac{2xy - x^2\frac{dy}{dx}}{y^2} + 1 = \frac{dy}{dx} \qquad \mathbf{2E1}$$

$$\frac{-6 - 9\frac{dy}{dx}}{1} + 1 = \frac{dy}{dx}$$

$$-6 - 9\frac{dy}{dx} + 1 = \frac{dy}{dx}$$

$$\boldsymbol{\frac{dy}{dx} = -\frac{1}{2}} \qquad \mathbf{2E1}$$

Mathod 3

$$\frac{x^2}{y} + x = y - 5 \Rightarrow x^2\left(\frac{1}{y}\right) + x = y - 5$$

$$2x\frac{1}{y} + x^2\left(-\frac{1}{y^2}\right)\frac{dy}{dx} + 1 = \frac{dy}{dx} \qquad \mathbf{2E1}$$

$$-6 - 9\frac{dy}{dx} + 1 = \frac{dy}{dx} \Rightarrow \boldsymbol{\frac{dy}{dx} = -\frac{1}{2}} \qquad \mathbf{2E1}$$

Note: a candidate may obtain $\frac{dy}{dx} = \frac{2xy+y^2}{y^2+x^2}$ (in 2 and 3) and then substitute.

2. (a)

$$\det\begin{pmatrix} t+4 & 3t \\ 3 & 5 \end{pmatrix} = 5(t+4) - 9t \qquad \mathbf{1}$$

$$= 20 - 4t$$

$$A^{-1} = \frac{1}{20-4t}\begin{pmatrix} 5 & -3t \\ -3 & t+4 \end{pmatrix} \qquad \mathbf{1,1}$$

(b)

$$20 - 4t = 0 \Rightarrow t = 5 \qquad \mathbf{1}$$

(c)

$$\begin{pmatrix} t+4 & 3 \\ 3t & 5 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 6 & 5 \end{pmatrix} \Rightarrow t = 2 \qquad \mathbf{1}$$

3.

$$e^y x^2 \frac{dy}{dx} = 1$$

$$e^y \frac{dy}{dx} = x^{-2}$$

$$\int e^y dy = \int x^{-2} dx \qquad \mathbf{1}$$

$$e^y = -x^{-1} + c \qquad \mathbf{1}$$

$y = 0$ when $x = 1$ so

$$1 = -1 + c \Rightarrow c = 2 \qquad \mathbf{1}$$

$$e^y = 2 - \frac{1}{x} \Rightarrow y = \ln\left(2 - \frac{1}{x}\right) \qquad \mathbf{1}$$

4. When $n = 1$, LHS = $\frac{1}{1 \times 2} = \frac{1}{2}$, RHS = $1 - \frac{1}{2} = \frac{1}{2}$. So true when $n = 1$. **1**

Assume true for $n = k$, $\sum_{r=1}^{k} \frac{1}{r(r+1)} = 1 - \frac{1}{k+1}$. **1**

Consider $n = k + 1$

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)} \qquad \mathbf{1}$$

$$= 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= 1 - \frac{k+2-1}{(k+1)(k+2)} = 1 - \frac{k+1}{(k+1)\big((k+1)+1\big)} \qquad \mathbf{1}$$

$$= 1 - \frac{1}{\big((k+1)+1\big)} \qquad \mathbf{1}$$

Thus, if true for $n = k$, statement is true for $n = k + 1$, and, since true for $n = 1$, true for all $n \geqslant 1$.

5. *Method 1*

$$\int_{\ln\frac{3}{2}}^{\ln 2} \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$$

Let $u = e^x - e^{-x}$, then $du = \left(e^x + e^{-x}\right)dx$. **1**

When $x = \ln\frac{3}{2}$, $u = \frac{3}{2} - \frac{2}{3} = \frac{5}{6}$ and when $x = \ln 2$, $u = 2 - \frac{1}{2} = \frac{3}{2}$. **1**

$$\int_{\ln\frac{3}{2}}^{\ln 2} \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \int_{5/6}^{3/2} \frac{du}{u}$$

$$= [\ln u]_{5/6}^{3/2}$$ **1**

$$= \ln\frac{3}{2} - \ln\frac{5}{6} = \ln\frac{9}{5}$$ **1**

Method 2

$$\int_{\ln\frac{3}{2}}^{\ln 2} \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx = \left[\ln\left(e^x - e^{-x}\right)\right]_{\ln\frac{3}{2}}^{\ln 2}$$ **1,1**

$$= \ln\left(2 - \frac{1}{2}\right) - \ln\left(\frac{3}{2} - \frac{2}{3}\right)$$ **1**

$$= \ln\frac{3}{2} - \ln\frac{5}{6} = \ln\frac{9}{5}$$ **1**

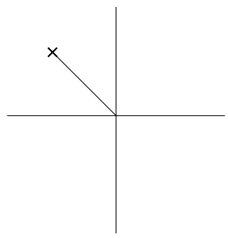
6.

$$\frac{(1 + 2i)^2}{7 - i} = \frac{1 + 4i - 4}{7 - i}$$ **1**

$$= \frac{-3 + 4i}{7 - i} \times \frac{7 + i}{7 + i}$$ **1**

$$= \frac{(-3 + 4i)(7 + i)}{50}$$

$$= -\frac{1}{2} + \frac{1}{2}i$$ **1**



1

$$|z| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{2}\sqrt{2}$$ **1**

$$\arg z = \tan^{-1}\frac{\frac{1}{2}}{-\frac{1}{2}} = \tan^{-1}(-1) = \frac{3\pi}{4}\ \left(\text{or } 135°\right).$$ **1**

7.

$$x = 2\sin\theta \Rightarrow dx = 2\cos\theta\, d\theta$$ **1**

$$x = 0 \Rightarrow \theta = 0;\ x = \sqrt{2} \Rightarrow \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$$ **1**

$$\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4 - x^2}}dx = \int_0^{\pi/4} \frac{4\sin^2\theta}{\sqrt{4 - 4\sin^2\theta}}\left(2\cos\theta\right) d\theta$$

$$= \int_0^{\pi/4} \frac{4\sin^2\theta}{2\cos\theta}\left(2\cos\theta\right) d\theta$$ **1**

$$= 2\int_0^{\pi/4} \left(2\sin^2\theta\right) d\theta$$

$$= 2\int_0^{\pi/4} \left(1 - \cos 2\theta\right) d\theta$$ **1**

$$= 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4}$$ **1**

$$= 2\left\{\left[\frac{\pi}{4} - \frac{1}{2}\right] - 0\right\}$$

$$= \frac{\pi}{2} - 1$$ **1**

8. (a)

$$(1 + x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$ **1**

(b) Let $x = -0{\cdot}1$, then **1**

$$0{\cdot}9^5 = (1 + (-0{\cdot}1))^5$$

$$= 1 - 0{\cdot}5 + 0{\cdot}1 - 0{\cdot}01 + 0{\cdot}0005 - 0{\cdot}00001$$ **1**

$$= 0{\cdot}5 + 0{\cdot}09 + 0{\cdot}00049$$

$$= 0{\cdot}59049$$

9.

$$\int_0^1 x\tan^{-1}x^2\,dx = \left[\tan^{-1}x^2 \int x\,dx\right]_0^1 - \int_0^1 \frac{2x}{1 + x^4}\,\frac{x^2}{2}dx$$ **1,1**

$$= \left[\frac{1}{2}x^2\tan^{-1}x^2\right]_0^1 - \int_0^1 \frac{x^3}{1 + x^4}dx$$

$$= \left[\frac{1}{2}x^2\tan^{-1}x^2\right]_0^1 - \left[\frac{1}{4}\ln\left(1 + x^4\right)\right]_0^1$$ **1**

$$= \frac{1}{2}\tan^{-1}1 - 0 - \left[\frac{1}{4}\ln 2 - \frac{1}{4}\ln 1\right]$$ **1**

$$= \frac{\pi}{8} - \frac{1}{4}\ln 2$$ **1,1**

10.

$$14654 = 11 \times 1326 + 68 \qquad \mathbf{1}$$
$$1326 = 19 \times 68 + 34$$
$$68 = 2 \times 34 \qquad \mathbf{1}$$
$$34 = 1326 - 19 \times 68 \qquad \mathbf{1}$$
$$= 1326 - 19(14654 - 11 \times 1326)$$
$$= 210 \times 1326 - 19 \times 14654 \qquad \mathbf{1}$$

11.

When $x = 1, y = 1.$ **1**

$$y = x^{2x^2 + 1}$$
$$\Rightarrow \ln y = \ln\left(x^{2x^2 + 1}\right) \qquad \mathbf{1}$$
$$= \left(2x^2 + 1\right) \ln x$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{2x^2 + 1}{x} + 4x \ln x \qquad \mathbf{1,1}$$

Hence, when $x = 1, y = 1$ and

$$\frac{dy}{dx} = 3 + 0 = 3. \qquad \mathbf{1}$$

12.

$$a_j = p^j \Rightarrow S_k = p + p^2 + \ldots + p^k = \frac{p\left(p^k - 1\right)}{p - 1}$$
$$S_n = \frac{p\left(p^n - 1\right)}{p - 1} \qquad \mathbf{1}$$
$$S_{2n} = \frac{p\left(p^{2n} - 1\right)}{p - 1} \qquad \mathbf{1}$$
$$\frac{p\left(p^{2n} - 1\right)}{p - 1} = \frac{65p\left(p^n - 1\right)}{p - 1}$$
$$\left(p^n + 1\right)\left(p^n - 1\right) = 65\left(p^n - 1\right) \qquad \mathbf{1}$$
$$p^n + 1 = 65 \qquad \mathbf{1}$$
$$\Rightarrow p^n = 64$$
$$a_2 = p^2 \Rightarrow a_3 = p^3 \text{ but } a_3 = 2p \text{ so } p^3 = 2p$$
$$\Rightarrow p^2 = 2 \Rightarrow p = \sqrt{2} \text{ since } p > 0. \qquad \mathbf{1}$$
$$p^n = 64 = 2^6 = \left(\sqrt{2}\right)^{12}$$
$$n = 12 \qquad \mathbf{1}$$

13.

$$f(x) = \frac{x^2 + 2x}{x^2 - 1} = \frac{x^2 + 2x}{(x - 1)(x + 1)}$$

Hence there are vertical asymptotes at $x = -1$ and $x = 1$. **1**

$$f(x) = \frac{x^2 + 2x}{x^2 - 1} = \frac{1 + \frac{2x}{x^2}}{1 - \frac{1}{x^2}} = \frac{1 + \frac{2}{x}}{1 - \frac{1}{x^2}}$$ **1**

$$\rightarrow 1 \text{ as } x \rightarrow \infty.$$

So $y = 1$ is a horizontal asymptote. **1**

$$f(x) = \frac{x^2 + 2x}{x^2 - 1}$$

$$f'(x) = \frac{(2x + 2)(x^2 - 1) - (x^2 + 2x)2x}{(x^2 - 1)^2}$$ **1**

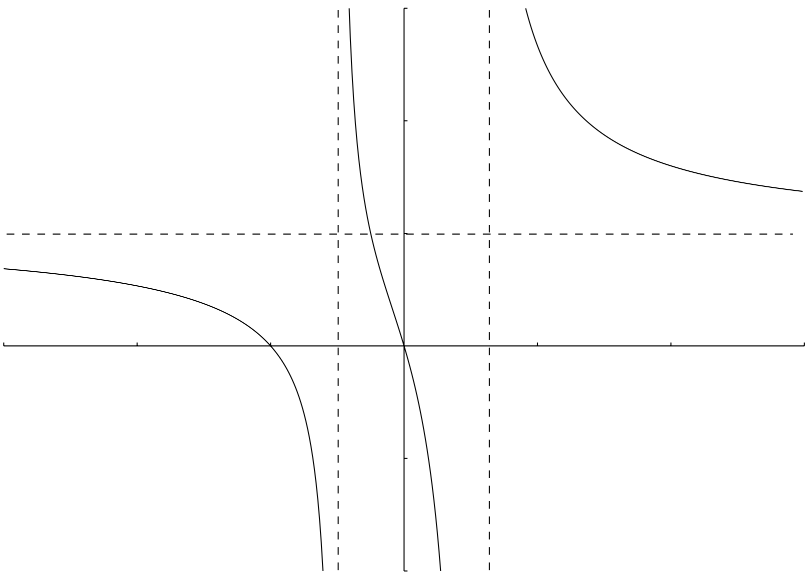
$$= \frac{2x^3 - 2x + 2x^2 - 2 - 2x^3 - 4x^2}{(x^2 - 1)^2} = \frac{-2(x^2 + x + 1)}{(x^2 - 1)^2}$$ **1**

$$= \frac{-2\left((x + \frac{1}{2})^2 + \frac{3}{4}\right)}{(x^2 - 1)^2} < 0$$ **1**

Hence $f(x)$ is a strictly decreasing function.

$$f(x) = \frac{x^2 + 2x}{x^2 - 1} = 0 \Rightarrow x = 0 \text{ or } x = -2$$ **1**

$$f(x) = \frac{x^2 + 2x}{x^2 - 1} = 1 \Rightarrow x^2 + 2x = x^2 - 1 \Rightarrow x = -\frac{1}{2}$$ **1**



1 for shape
1 for position

Alternatively for the horizontal asymptote:

$$\begin{array}{r|l} & \quad\quad\quad 1 \\ x^2 - 1 & x^2 + 2x \\ & x^2 \quad\quad - 1 \\ \hline & \quad 2x + 1 \end{array} \Rightarrow f(x) = 1 + \frac{2x + 1}{x^2 - 1} \rightarrow 1 \text{ as } x \rightarrow \infty$$

14.

$$\frac{x^2+6x-4}{(x+2)^2(x-4)} = \frac{A}{(x+2)^2} + \frac{B}{x+2} + \frac{C}{x-4}$$ **M1**

$$x^2+6x-4 = A(x-4) + B(x+2)(x-4) + C(x+2)^2$$

Let $x = -2$ then $4 - 12 - 4 = -6A \Rightarrow A = 2.$ **1**

Let $x = 4$ then $16 + 24 - 4 = 36C \Rightarrow C = 1.$ **1**

Let $x = 0$ then

$-4 = -4A - 8B + 4C \Rightarrow -4 = -8 - 8B + 4 \Rightarrow B = 0.$ **1**

Thus

$$\frac{x^2+6x-4}{(x+2)^2(x-4)} = \frac{2}{(x+2)^2} + \frac{1}{x-4}.$$

Let $f(x) = 2(x+2)^{-2} + (x-4)^{-1}$ then

$f(x) = 2(x+2)^{-2} + (x-4)^{-1} \Rightarrow f(0) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ **1**

$f'(x) = -4(x+2)^{-3} - (x-4)^{-2} \Rightarrow f'(0) = -\frac{1}{2} - \frac{1}{16} = -\frac{9}{16}$ **1**

$f''(x) = 12(x+2)^{-4} + 2(x-4)^{-3} \Rightarrow f''(0) = \frac{3}{4} - \frac{1}{32} = \frac{23}{32}$ **1**

Thus

$$\frac{x^2+6x-4}{(x+2)^2(x-4)} = \frac{1}{4} - \frac{9x}{16} + \frac{23x^2}{64} + \ldots$$ **2E1**

15. (a)

$$(x + 1)\frac{dy}{dx} - 3y = (x + 1)^4$$

$$\frac{dy}{dx} - \frac{3}{x + 1}y = (x + 1)^3 \qquad \mathbf{1}$$

Integrating factor:

$$\text{since} \int \frac{-3}{x + 1}dx = -3\ln(x + 1). \qquad \mathbf{1}$$

Hence the integrating factor is $(x + 1)^{-3}$. **1**

$$\frac{1}{(x + 1)^3}\frac{dy}{dx} - \frac{3}{(x + 1)^4}y = 1 \qquad \mathbf{1}$$

$$\frac{d}{dx}\left((x + 1)^{-3}y\right) = 1$$

$$\frac{y}{(x + 1)^3} = \int 1\, dx \qquad \mathbf{1}$$

$$= x + c$$

$y = 16$ when $x = 1$, so $2 = 1 + c \Rightarrow c = 1$. Hence

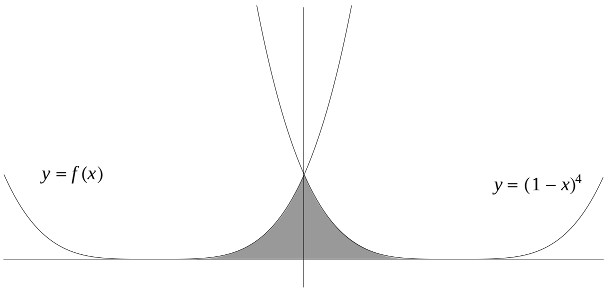
$$y = (x + 1)^4 \qquad \mathbf{1}$$

(b)

$$(x + 1)^4 = (1 - x)^4$$

$$x + 1 = 1 - x \Rightarrow x = 0 \qquad \mathbf{1}$$

or $x + 1 = -1 + x$ which has no solutions.



$$\text{Area} = \int_{-1}^{0} (x + 1)^4\, dx + \int_{0}^{1} (1 - x)^4\, dx \qquad \mathbf{M1}$$

$$= 2\int_{-1}^{0} (x + 1)^4\, dx \qquad \mathbf{1}$$

$$= \frac{2}{5}\left[(x + 1)^5\right]_{-1}^{0} = \frac{2}{5} - 0 = \frac{2}{5} \qquad \mathbf{1}$$

16. (a)

$$x + y - z = 6$$
$$2x - 3y + 2z = 2$$
$$-5x + 2y - 4z = 1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 2 & -3 & 2 & 2 \\ -5 & 2 & -4 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & -5 & 4 & -10 \\ 0 & 7 & -9 & 31 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & -5 & 4 & -10 \\ 0 & 0 & -\frac{17}{5} & 17 \end{array}\right)$$

1,1,1

$$z = 17 \div \left(\frac{-17}{5}\right) = -5$$ **1**

$$-5y - 20 = -10 \Rightarrow y = -2$$

$$x - 2 + 5 = 6 \Rightarrow x = 3$$ **1**

(b) Let $x = \lambda$.

Method 1

In first plane: $x + y - z = 6$.

$\lambda + (4\lambda - 14) - (5\lambda - 20) = 5\lambda - 5\lambda + 6 = 6.$ **1**

In the second plane:

$2x - 3y + 2z = 2\lambda - 3(4\lambda - 14) + 2(5\lambda - 20) = 5\lambda - 5\lambda + 2 = 2.$ **1**

Method 2

$$y - z = 6 - \lambda \Rightarrow y = 6 + z - \lambda$$
$$-3y + 2z = 2 - 2\lambda$$ **1**
$$(-18 - 3z + 3\lambda) + 2z = 2 - 2\lambda$$
$$-z = 20 - 5\lambda \Rightarrow z = 5\lambda - 20$$ **1**
$$\text{and } y = 4\lambda - 14$$

Method 2

$x + y - z =$	6	(1)
$2x - 3y + 2z =$	2	(2)
$5x - z =$	20	(2) + 3(1)
$4x - y =$	14	(2) + 2(1)

1

$$y = 4x - 14$$
$$z = 5x - 20$$

$$x = \lambda,\ y = 4\lambda - 14,\ z = 5\lambda - 20$$ **1**

(c) Direction of L is $\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$, direction of normal to the plane is $-5\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$. Letting θ be the angle between these then

$$\cos\theta = \frac{-5 + 8 - 20}{\sqrt{42}\sqrt{45}}$$ **1M,1**

$$= \frac{-17}{3\sqrt{210}}$$

This gives a value of 113.0° which leads to the angle $113.0° - 90° = 23.0°$. **1,1**