2011 Maths

Advanced Higher

Finalised Marking Instructions

General Marking Principles

These principles describe the approach taken when marking Advanced Higher Mathematics papers. For more detailed guidance please refer to the detailed Marking Instructions.

1 The main principle is to give credit for the skills demonstrated and the criteria met. Failure to have a correct method may not preclude a candidate gaining credit for their solution.

2 The answer to one part of a question, even if incorrect, can be accepted as a basis for subsequent dependent parts of the question.

3 The following are not penalised:

- working subsequent to a correct answer (unless it provides firm evidence that the requirements of the question have not been met)
- legitimate variation in numerical values / algebraic expressions.

4 Full credit will only be given where the solution contains appropriate working. Where the correct answer might be obtained by inspection or mentally, credit may be given.

5 Sometimes the method to be used in a particular question is explicitly stated; no credit will be given where a candidate obtains the correct answer by an alternative method.

6 Where the method to be used in a particular question is not explicitly stated in the question paper, full credit is available for an alternative valid method. (Some likely alternatives are included but these should not be assumed to be the only acceptable ones.)

In the detailed Marking Instructions which follow, marks are shown alongside the line for which they are awarded. There is one code used M. This indicates a method mark, so in question 1(a), 1M means a method mark for the product rule.

Advanced Higher Mathematics 2011

			Marks awarded for
1. (5)	$\dfrac{13-x}{x^2+4x-5} = \dfrac{13-x}{(x-1)(x+5)}$		
	$= \dfrac{A}{x-1} + \dfrac{B}{x+5}$	**1**	
	$13 - x = A(x+5) + B(x-1)$		
	$x = 1 \Rightarrow 12 = 6A \Rightarrow A = 2$	**1**	for first value
	$x = -5 \Rightarrow 18 = -6B \Rightarrow B = -3$	**1**	for second value
	Hence $\dfrac{13-x}{x^2+4x-5} = \dfrac{2}{x-1} - \dfrac{3}{x+5}$		
	$\displaystyle\int \frac{13-x}{x^2+4x-5}dx = \int \frac{2}{x-1}dx - \int \frac{3}{x+5}dx$		
	$= 2\ln\lvert x-1\rvert - 3\ln\lvert x+5\rvert + c$	**1**	for logs
		1	for moduli
2. (3)	$\left(\tfrac{1}{2}x - 3\right)^4 = {}^4C_0\left(\tfrac{x}{2}\right)^4 + {}^4C_1\left(\tfrac{x}{2}\right)^3(-3) + {}^4C_2\left(\tfrac{x}{2}\right)^2(-3)^2 + {}^4C_3\left(\tfrac{x}{2}\right)(-3)^3 + {}^4C_4(-3)^4$	**1**	for powers
		1	for coefficients
	$= \left(\tfrac{x}{2}\right)^4 + 4\left(\tfrac{x}{2}\right)^3(-3) + 6\left(\tfrac{x}{2}\right)^2(-3)^2 + 4\left(\tfrac{x}{2}\right)(-3)^3 + (-3)^4$		
	$= \dfrac{x^4}{16} - \dfrac{3x^3}{2} + \dfrac{27x^2}{2} - 54x + 81.$	**1**	for simplifying
3. (6)	(a) *Method 1*		
	$y + e^y = x^2$		
	$\dfrac{dy}{dx} + e^y\dfrac{dy}{dx} = 2x$	**1M**	for applying implicit differentiation
		1	for accuracy
	$\dfrac{dy}{dx}\left(1 + e^y\right) = 2x \Rightarrow \dfrac{dy}{dx} = \dfrac{2x}{\left(1 + e^y\right)}$	**1**	
	Method 2		
	$\ln\left(y + e^y\right) = 2\ln x$	**1**	
	$\dfrac{(1+e^y)\frac{dy}{dx}}{y + e^y} = \dfrac{2}{x}$	**1**	
	$\dfrac{dy}{dx} = \dfrac{2\left(y + e^y\right)}{x\left(1 + e^y\right)}$	**1**	
	Method 3		
	$y + e^y = x^2 \Rightarrow e^y = x^2 - y \Rightarrow y = \ln\left(x^2 - y\right)$	**1**	
	$\dfrac{dy}{dx} = \dfrac{2x - \frac{dy}{dx}}{x^2 - y}$	**1**	
	$\frac{dy}{dx}\left(x^2 - y\right) = 2x - \frac{dy}{dx} \Rightarrow \frac{dy}{dx}\left(x^2 - y + 1\right) = 2x$		
	$\dfrac{dy}{dx} = \dfrac{2x}{x^2 - y + 1}$	**1**	

3.	(b) *Method 1*		
	$f(x) = \sin x \cos^3 x$		
		M1	for using the product rule
	$f'(x) = \cos^4 x + \sin x\left(-3\cos^2 x \sin x\right)$	**1**	for first term
		1	for second term
	$= \cos^4 x - 3\cos^2 x \sin^2 x$		
	Method 2		
	$f(x) = \sin x \cos^3 x$		
	$\ln\left(f(x)\right) = \ln \sin x + \ln\left(\cos^3 x\right)$	**M1**	
	$\dfrac{f'(x)}{f(x)} = \dfrac{\cos x}{\sin x} - \dfrac{3\cos^2 x \sin x}{\cos^3 x}$	**1**	
	$= \dfrac{\cos x}{\sin x} - \dfrac{3\sin x}{\cos x}$		
	$f'(x) = \left(\dfrac{\cos x}{\sin x} - \dfrac{3\sin x}{\cos x}\right)\sin x \cos^3 x$	**1**	
	$= \cos^4 x - 3\sin^2 x \cos^2 x$		

4. **(6)**	(a) Singular when the determinant is 0.		
	$1\det\begin{pmatrix} 0 & 2 \\ \lambda & 6 \end{pmatrix} - 2\det\begin{pmatrix} 3 & 2 \\ -1 & 6 \end{pmatrix} + (-1)\det\begin{pmatrix} 3 & 0 \\ -1 & \lambda \end{pmatrix} = 0$	**M1**	
	$-2\lambda - 2(18 + 2) - 1(3\lambda - 0) = 0$	**1**	
	$-5\lambda - 40 = 0$ when $\lambda = -8$	**1**	
	(b) From A, $A' = \begin{pmatrix} 2 & 3\alpha + 2\beta & -1 \\ 2\alpha - \beta & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix}$.	**1**	for transpose
	Hence $2\alpha - \beta = -1$ and $3\alpha + 2\beta = -5$.	**1**	
	$4\alpha - 2\beta = -2$		
	$3\alpha + 2\beta = -5$		
	$7\alpha = -7 \Rightarrow \alpha = -1$ and $\beta = -1$.	**1**	for both values

Marks awarded for

5. (6)

Let $f(x) = (1+x)^{\frac{1}{2}}$, then

$$f(x) = (1+x)^{\frac{1}{2}} \Rightarrow f(0) = 1$$

$$f'(x) = \tfrac{1}{2}(1+x)^{-\frac{1}{2}} \Rightarrow f'(0) = \tfrac{1}{2}$$

$$f''(x) = -\tfrac{1}{4}(1+x)^{-\frac{3}{2}} \Rightarrow f''(0) = -\tfrac{1}{4}$$

$$f'''(x) = \tfrac{3}{8}(1+x)^{-\frac{5}{2}} \Rightarrow f'''(0) = \tfrac{3}{8}$$

1 for derivatives
1 for values

Hence

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{4} \times \frac{x^2}{2} + \frac{3}{8} \times \frac{x^3}{6} - \ldots$$ **1**

$$= 1 + \frac{1}{2}x - \frac{x^2}{8} + \frac{x^3}{16} - \ldots$$

and replacing x by x^2 gives

$$\sqrt{1+x^2} = 1 + \frac{1}{2}x^2 - \frac{x^4}{8} + \frac{x^6}{16} - \ldots$$ **1**

Thus

$$\sqrt{(1+x)(1+x^2)} =$$

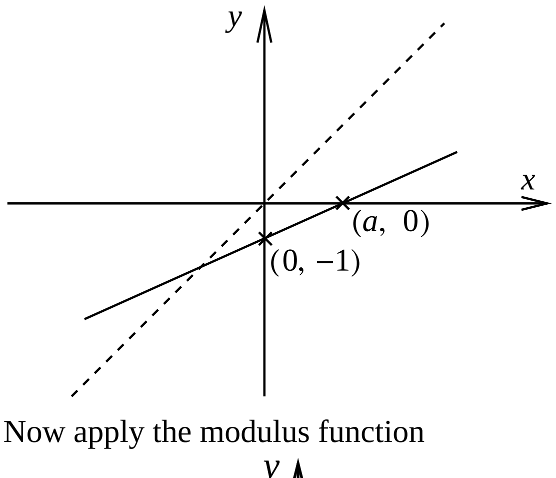
$$\left(1 + \frac{1}{2}x - \frac{x^2}{8} + \frac{x^3}{16} - \ldots\right)\left(1 + \frac{1}{2}x^2 - \frac{x^4}{8} + \frac{x^6}{16} - \ldots\right)$$ **1M** for multiplying

$$= 1 + \frac{1}{2}x + \frac{1}{2}x^2 - \frac{1}{8}x^2 + \frac{1}{4}x^3 + \frac{1}{16}x^3 + \ldots$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots$$ **1**

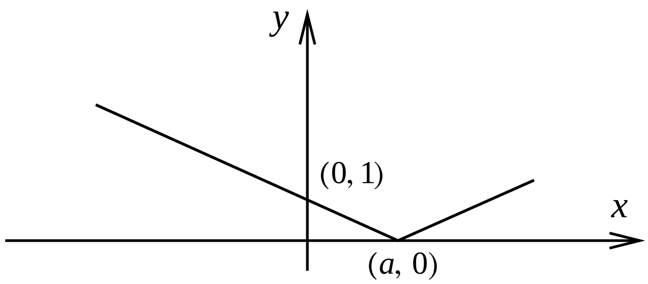
6. (4)

Reflect in the line $y = x$ to get



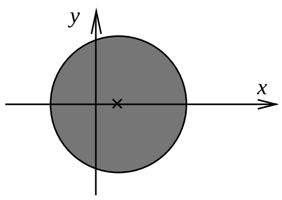
1 for position
1 for coordinates

Now apply the modulus function



1 for shape
1 for coordinates

7. (4)	*Method 1*		
	$y = \dfrac{e^{\sin x}(2+x)^3}{\sqrt{1-x}}$		
	$\Rightarrow \ln y = \ln\left(e^{\sin x}(2+x)^3\right) - \ln\left(\sqrt{1-x}\right)$	**1M**	for use of logs
	$= \sin x + 3\ln(2+x) - \frac{1}{2}\ln(1-x)$	**1**	for preparing to differentiate
	$\Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = \cos x + \dfrac{3}{2+x} + \dfrac{1}{2(1-x)}$	**1**	
	$\dfrac{dy}{dx} = y\left(\cos x + \dfrac{3}{2+x} + \dfrac{1}{2(1-x)}\right)$		
	When $x = 0, y = 8 \Rightarrow$		
	gradient $= 8\left(1 + \dfrac{3}{2} + \dfrac{1}{2}\right) = 24.$	**1**	for final value
	Method 2		
	$y = \dfrac{e^{\sin x}(2+x)^3}{\sqrt{1-x}} \Rightarrow \dfrac{dy}{dx} =$		
	$\dfrac{\frac{d}{dx}\left(e^{\sin x}(2+x)^3\right)\sqrt{1-x} - e^{\sin x}(2+x)^3\left(-\frac{1}{2}\frac{1}{\sqrt{1-x}}\right)}{(1-x)}$	**M1**	for use of quotient rule
	$= \dfrac{\left[\cos x\, e^{\sin x}(2+x)^3 + 3e^{\sin x}(2+x)^2\right](1-x)}{(1-x)^{3/2}}$	**M1**	
	$+ \dfrac{e^{\sin x}(2+x)^3}{2(1-x)^{3/2}}$	**1**	
	When $x = 0$,		
	gradient $= \dfrac{(2^3 + 3\times 2^2)}{1} + \dfrac{2^3}{2} = 20 + 4 = 24$	**1**	
	Method 3		
	$y = \frac{e^{\sin x}(2+x)^3}{\sqrt{1-x}}$		
	$y\sqrt{1-x} = e^{\sin x}(2+x)^3$	**1**	
	$\sqrt{1-x}\frac{dy}{dx} - \frac{1}{2}y(1-x)^{-1/2}$ $= \cos x\, e^{\sin x}(2+x)^3 + 3e^{\sin x}(2+x)^2$	**1,1**	
	when $x = 0, y = \frac{e^0\, 2^3}{1} = 8.$ This leads to		
	$\frac{dy}{dx} = 24$	**1**	

			Marks awarded for
8. **(4)**	$\sum_{r=1}^{n} r^3 - \left(\sum_{r=1}^{n} r\right)^2 = \frac{n^2(n+1)^2}{4} - \left(\frac{n(n+1)}{2}\right)^2 = 0$	**1**	
	$\sum_{r=1}^{n} r^3 + \left(\sum_{r=1}^{n} r\right)^2 = \frac{n^2(n+1)^2}{4} + \left(\frac{n(n+1)}{2}\right)^2$	**1**	
	$= \frac{n^2(n+1)^2}{4} + \frac{n^2(n+1)^2}{4}$	**1**	
	$= \frac{n^2(n+1)^2}{2}$	**1**	
9. **(5)**	*Method 1*		
	$\frac{dy}{dx} = 3(1+y)\sqrt{1+x}$		
	$\int \frac{dy}{1+y} = 3\int (1+x)^{\frac{1}{2}}\,dx$	**M1**	separating variables
	$\ln(1+y) = 2(1+x)^{\frac{3}{2}} + c$	**1** **1**	for LHS for term in x
	$1 + y = \exp\left(2(1+x)^{\frac{3}{2}} + c\right)$	**1**	for the constant
	$y = \exp\left(2(1+x)^{\frac{3}{2}} + c\right) - 1.$	**1**	
	$= A\exp\left(2(1+x)^{\frac{3}{2}}\right) - 1.$		
	Method 2		
	$\frac{dy}{dx} - 3\sqrt{1+x}\,y = 3\sqrt{1+x}$	**1**	
	Integrating Factor		
	$\exp\left(-3\int\sqrt{1+x}\,dx\right) = \exp\left(-2(1+x)^{3/2}\right)$	**1**	
	$\frac{d}{dx}\left(y\exp\left(-2(1+x)^{3/2}\right)\right) =$ $3\sqrt{1+x}\left(\exp\left(-2(1+x)^{3/2}\right)\right)$	**1**	
	$y\left(\exp\left(-2(1+x)^{3/2}\right)\right) =$ $-\int \left(-3\sqrt{1+x}\right)\exp\left(-2(1+x)^{3/2}\right)dx$		
	$= -\exp\left(-2(1+x)^{3/2}\right) + c$	**1**	
	$y = -1 + c\exp\left(2(1+x)^{3/2}\right)$	**1**	
10. **(5)**	Let $z = x + iy$, so		
	$z - 1 = (x-1) + iy.$	**1**	
	$\lvert z-1\rvert^2 = (x-1)^2 + y^2 = 9.$	**1**	
	The locus is the circle with centre $(1, 0)$ and radius 3.	**1**	Can subsume the first two marks.
	 	**1** **1**	for circle for shading or other indication

11. **(7)**	(a) $\int_0^{\pi/4}(\sec x - x)(\sec x + x)\,dx = \int_0^{\pi/4}(\sec^2 x - x^2)\,dx$	**1**	
	$= \left[\tan x - \dfrac{x^3}{3}\right]_0^{\frac{\pi}{4}}$	**1**	
	$= \left[1 - \dfrac{1}{3}\dfrac{\pi^3}{64}\right] - [0]$		
	$= 1 - \dfrac{\pi^3}{192}.$	**1**	Exact value only
	(b) *Method 1*		
	Let $u = 7x^2$,	**M1**	
	then $du = 14x\,dx$.	**1**	
	$\int \dfrac{x}{\sqrt{1-49x^4}}\,dx = \dfrac{1}{14}\int \dfrac{du}{\sqrt{1-u^2}}$	**1**	
	$= \dfrac{1}{14}\sin^{-1} u + c$		
	$= \dfrac{1}{14}\sin^{-1} 7x^2 + c$	**1**	must be in terms of x
	Method 2		
		1	for fraction
	$\int \dfrac{x}{\sqrt{1-49x^4}}\,dx = \dfrac{1}{14}\int \dfrac{14x\,dx}{\sqrt{1-(7x^2)^2}}$	**1**	for numerator
		1	for $(7x^2)^2$
	$= \dfrac{1}{14}\sin^{-1} 7x^2 + c$	**1**	must be in terms of x
12. **(5)**	For $n = 2$, $8^2 + 3^0 = 64 + 1 = 65$.		
	True when $n = 2$.	**1**	
	Assume true for k, i.e. that $8^k + 3^{k-2}$ is divisible by 5, i.e. can be expressed as $5p$ for an integer p.	**1**	for the inductive hypothesis
	Now consider $8^{k+1} + 3^{k-1}$		
	$= 8 \times 8^k + 3^{k-1}$	**1**	
	$= 8 \times (5p - 3^{k-2}) + 3^{k-1}$	**1**	for replacing 8^k
	$= 40p - 3^{k-2}(8 - 3)$		
	$= 5(8p - 3^{k-2})$ which is divisible by 5.	**1**	
	So, since it is true for $n = 2$, it is true for all $n \geqslant 2$.		

13.
(9)

Method 1

Let d be the common difference. Then

$u_3 = 1 = a + 2d$ and $u_2 = \frac{1}{a} = a + d$ **1**

$1 = a + 2\left(\frac{1}{a} - a\right)$ **1**

$a = a^2 + 2 - 2a^2$

$a^2 + a - 2 = 0$ **1**

$(a+2)(a-1) = 0 \Rightarrow a = -2$ since $a < 0$. **1**

$a = -2$ gives $2d = 3$ and hence $d = \frac{3}{2}$. **1**

Method 2

$u_1 = a, u_2 = \frac{1}{a}, u_3 = 1$ **M1**

$\Rightarrow \frac{1}{a} - a = 1 - \frac{1}{a}$ **1**

$\Rightarrow 1 - a^2 = a - 1$

$\Rightarrow a^2 + a - 2 = 0$ **1**

$(a+2)(a-1) = 0 \Rightarrow a = -2$ since $a < 0$. **1**

$d = u_3 - u_2 = 1 - \frac{1}{a} = \frac{3}{2}$ **1**

$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$= \frac{n}{2}\left[-4 + \frac{3}{2}n - \frac{3}{2}\right]$ **1**

$= \frac{1}{4}\left[3n^2 - 11n\right]$

$\therefore \quad 3n^2 - 11n > 4000$ **1**

$n^2 - \frac{11}{3}n > \frac{4000}{3}$

$\left(n - \frac{11}{6}\right)^2 > \frac{48000}{36} + \frac{121}{36} = \frac{48121}{36}$

$n - \frac{11}{6} > \frac{\sqrt{48121}}{6}$

$n > \frac{\sqrt{48121} + 11}{6} \approx 38.39$

So the least value of n is 39. **1** for value
1 for suitable justification

			Marks awarded for
14.	Auxiliary equation		
(10)	$m^2 - m - 2 = 0$	**1**	
	$(m - 2)(m + 1) = 0$		
	$m = -1$ or 2	**1**	
	Complementary function is: $y = Ae^{-x} + Be^{2x}$	**1**	
	The particular integral has the form $y = Ce^x + D$	**1**	
	$y = Ce^x + D \Rightarrow \dfrac{dy}{dx} = Ce^x$		
	$\Rightarrow \dfrac{d^2y}{dx^2} = Ce^x$	**1**	
	Hence we need:		
	$\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = e^x + 12$		
	$\left[Ce^x\right] - \left[Ce^x\right] - 2\left[Ce^x + D\right] = e^x + 12$		
	$-2Ce^x - 2D = e^x + 12$	**1**	
	Hence $C = -\frac{1}{2}$ and $D = -6$.		
	So the General Solution is		
	$y = Ae^{-x} + Be^{2x} - \frac{1}{2}e^x - 6.$	**1**	
	$x = 0$ and $y = -\frac{3}{2} \Rightarrow$		
	$A + B - \frac{1}{2} - 6 = -\frac{3}{2}$		
	$x = 0$ and $\frac{dy}{dx} = \frac{1}{2} \Rightarrow$	**1**	Setting up the equations
	$-A + 2B - \frac{1}{2} = \frac{1}{2}$		
	$3B - 7 = -1 \Rightarrow B = 2 \Rightarrow A = 3$	**1**	
	So the particular solution is		
	$y = 3e^{-x} + 2e^{2x} - \frac{1}{2}e^x - 6.$	**1**	

			Marks awarded for
15. (10)	(a) In terms of a parameter s, L_1 is given by $x = 1 + ks,\ y = -s,\ z = -3 + s$	**1**	
	In terms of a parameter t, L_2 is given by $x = 4 + t, y = -3 + t, z = -3 + 2t$	**1**	
	Equating the y coordinates and equating the z coordinates: $\left.\begin{array}{r} -s = -3 + t \\ -3 + s = -3 + 2t \end{array}\right\}$	**1**	
	Adding these $-3 = -6 + 3t$ $\Rightarrow t = 1 \Rightarrow s = 2.$	**1**	
	From the x coordinates $1 + ks = 4 + t$ Using the values of s and t $1 + 2k = 5 \Rightarrow k = 2$	**1**	
	The point of intersection is: $(5, -2, -1)$.	**1**	
	(b) L_1 has direction $2\mathbf{i} - \mathbf{j} + \mathbf{k}$. L_2 has direction $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.	**1**	For both directions.
	Let the angle between L_1 and L_2 be θ, then $\cos\theta = \dfrac{(2\mathbf{i} - \mathbf{j} + \mathbf{k}).(\mathbf{i} + \mathbf{j} + 2\mathbf{k})}{\lvert 2\mathbf{i} - \mathbf{j} + \mathbf{k}\rvert \lvert \mathbf{i} + \mathbf{j} + 2\mathbf{k}\rvert}$	**1**	
	$= \dfrac{2 - 1 + 2}{\sqrt{6}\sqrt{6}} = \dfrac{3}{6} = \dfrac{1}{2}$	**1**	
	$\theta = 60°$	**1**	
	The angle between L_1 and L_2 is 60°.		

16.
(11)

(a) $I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,dx$

$= \int_0^1 1 \times \left(1+x^2\right)^{-n}\,dx$ **1** for showing that 1 is integrated

$= \left[(1+x^2)^{-n}\int 1\,dx\right]_0^1 + \int_0^1\left(2nx(1+x^2)^{-n-1}\int 1\,dx\right)dx$ **1**

$= \left[x(1+x^2)^{-n}\right]_0^1 + \int_0^1 2nx^2(1+x^2)^{-n-1}\,dx$

$= \frac{1}{2^n} - 0 + 2n\int_0^1 x^2\left(1+x^2\right)^{-n-1}\,dx$ **1**

$= \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx.$

(b) $\frac{A}{(1+x^2)^n} + \frac{B}{(1+x^2)^{n+1}} = \frac{x^2}{(1+x^2)^{n+1}}$

$\Rightarrow A\left(1+x^2\right) + B = x^2$ **1**

$\Rightarrow A = 1,\ B = -1$ **1**

$\frac{1}{(1+x^2)^n} + \frac{-1}{(1+x^2)^{n+1}} = \frac{x^2}{(1+x^2)^{n+1}}$ (*)

$I_n = \frac{1}{2^n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx.$

$= \frac{1}{2^n} + 2n\int_0^1 \frac{1}{(1+x^2)^n}\,dx + 2n\int_0^1 \frac{-1}{(1+x^2)^{n+1}}\,dx$ **1** Using (*)

$= \frac{1}{2^n} + 2nI_n - 2nI_{n+1}$ **1** Recognising I_n and I_{n+1}

$2nI_{n+1} = \frac{1}{2^n} + (2n-1)\,I_n$ **1**

$I_{n+1} = \frac{1}{n\times 2^{n+1}} + \left(\frac{2n-1}{2n}\right)I_n.$

(c) $\int_0^1 \frac{1}{(1+x^2)^3}\,dx = I_3$

$= \frac{1}{16} + \frac{3}{4}I_2$ **1**

$= \frac{1}{16} + \frac{3}{4}\left(\frac{1}{4} + \frac{1}{2}I_1\right)$

$= \frac{1}{4} + \frac{3}{8}\int_0^1 \frac{1}{1+x^2}dx$

$= \frac{1}{4} + \frac{3}{8}\left[\tan^{-1}x\right]_0^1$ **1**

$= \frac{1}{4} + \frac{3}{8}\frac{\pi}{4} = \frac{1}{4} + \frac{3\pi}{32}.$ **1**

END OF SOLUTIONS