2013 Mathematics

Advanced Higher

Finalised Marking Instructions

Part One: General Marking Principles for Mathematics Advanced Higher

This information is provided to help you understand the general principles you must apply when marking candidate responses to questions in this Paper. These principles must be read in conjunction with the specific Marking Instructions for each question.

(a) Marks for each candidate response must always be assigned in line with these general marking principles and the specific Marking Instructions for the relevant question. If a specific candidate response does not seem to be covered by either the principles or detailed Marking Instructions, and you are uncertain how to assess it, you must seek guidance from your Principal Assessor.

(b) Marking should always be positive i.e, marks should be awarded for what is correct and not deducted for errors or omissions.

GENERAL MARKING ADVICE: Mathematics Advanced Higher

The marking schemes are written to assist in determining the "minimal acceptable answer" rather than listing every possible correct and incorrect answer. The following notes are offered to support Markers in making judgements on candidates' evidence, and apply to marking both end of unit assessments and course assessments.

General Marking Principles

These principles describe the approach taken when marking Advanced Higher Mathematics papers. For more detailed guidance please refer to the detailed Marking Instructions.

1 The main principle is to give credit for the skills demonstrated and the criteria met. Failure to have a correct method may not preclude a candidate gaining credit for their solution.

2 The answer to one part of a question, even if incorrect, can be accepted as a basis for subsequent dependent parts of the question.

3 The following are not penalised:

- working subsequent to a correct answer (unless it provides firm evidence that the requirements of the question have not been met)
- legitimate variation in numerical values / algebraic expressions.

4 Full credit will only be given where the solution contains appropriate working. Where the correct answer might be obtained by inspection or mentally, credit may be given.

5 Sometimes the method to be used in a particular question is explicitly stated; no credit will be given where a candidate obtains the correct answer by an alternative method.

6 Where the method to be used in a particular question is not explicitly stated in the question paper, full credit is available for an alternative valid method. (Some likely alternatives are included but these should not be assumed to be the only acceptable ones.)

In the detailed Marking Instructions which follow, marks are shown alongside the line for which they are awarded. When marking, no comments at all should be made on the script. The total mark for each question should appear in one of the right-hand margins. The following codes should be used where applicable:

√ - correct; X – wrong; working underlined – wrong;

tickcross – mark(s) awarded for follow-through from previous answer;

^ ^ - mark(s) lost through omission of essential working or incomplete answer;

wavy or broken underline – bad form, but not penalised.

Part Two: Marking Instructions for each Question

Question	Expected Answer/s	Max Mark	Additional Guidance
1.	**Write down the binomial expansion of** $\left(3x-\dfrac{2}{x^2}\right)^4$ **and simplify your answer.**	**4**	
	${}^4C_0(3x)^4\left(\dfrac{-2}{x^2}\right)^0+{}^4C_1(3x)^3\left(\dfrac{-2}{x^2}\right)^1+{}^4C_2(3x)^2\left(\dfrac{-2}{x^2}\right)^2$ $+{}^4C_3(3x)^1\left(\dfrac{-2}{x^2}\right)^3+{}^4C_4(3x)^0\left(\dfrac{-2}{x^2}\right)^4$		•¹ Correct binomial coefficients.[2] •² Correct powers of $3x$ and $\frac{-2}{x^2}$.
	$=81x^4+4\cdot 27x^3\cdot\dfrac{-2}{x^2}+6\cdot 9x^2\cdot\dfrac{4}{x^4}+4\cdot 3x\cdot\dfrac{-8}{x^6}+\dfrac{16}{x^8}$ $=81x^4-216x+\dfrac{216}{x^2}-\dfrac{96}{x^5}+\dfrac{16}{x^8}$		•³ Simplifies indices.[1] •⁴ Completes simplification of coefficients.[3]

Notes:

1.1 Accept negative indices.

1.2 Award •¹ nCr or $\binom{n}{r}$ form.

1.3 Including signs. " + -" or "- -" : do not award •⁴

1.4 Expanding wrong expression: $\left(3x-\frac{2}{x}\right)^4$, •¹ •⁴ only are available.

1.5 Expanding $\left(3x+\frac{2}{x^2}\right)^4$, •¹ •³ •⁴ only are available.

Question	Expected Answer/s	Max Mark	Additional Guidance
2.	**Differentiate** $f(x) = e^{\cos x}\sin^2 x$**.**	**3**	
	$f'(x) = e^{\cos x}(-\sin x).\sin^2 x + e^{\cos x}.2\sin x\cos x$		•¹ Uses product rule.[1] •² First term correct. •³ Second term correct.[2] Simplified alternatives.
	$= -e^{\cos x}\sin^3 x + e^{\cos x}.2\sin x\cos x$ $= e^{\cos x}(\sin 2x - \sin^3 x)$ $= e^{\cos x}\sin x(2\cos x - \sin^2 x)$		

Notes:

2.1 Evidence of method: Statement of the rule and evidence of progress in applying it.
OR Application showing the ***sum*** of two terms, both involving differentiation.

2.2 Signs switched: •¹ •³ available for $e^{\cos x}\sin^3 x - e^{\cos x}.2\sin x\cos x$ or equivalent.

<table>
<tr><th colspan="3">Question</th><th>Expected Answer/s</th><th>Max Mark</th><th>Additional Guidance</th></tr>
<tr><td>3.</td><td></td><td></td><td>Matrices A and B are defined by $A=\begin{pmatrix}4 & p\\ -2 & 1\end{pmatrix}$
and $B=\begin{pmatrix}x & -6\\ 1 & 3\end{pmatrix}$
(a) Find A^2.
(b) Find the value of p for which A^2 is singular.
(c) Find the values of p and x if $B = 3A'$.</td><td>

1
2
2</td><td></td></tr>
<tr><td></td><td>a</td><td></td><td>$A^2=\begin{pmatrix}4 & p\\ -2 & 1\end{pmatrix}\begin{pmatrix}4 & p\\ -2 & 1\end{pmatrix}=\begin{pmatrix}16-2p & 4p+p\\ -8-2 & -2p+1\end{pmatrix}$
$=\begin{pmatrix}16-2p & 5p\\ -10 & 1-2p\end{pmatrix}$</td><td></td><td>•[1] Correct answer.[3,1]

Improved alternative.</td></tr>
<tr><td></td><td>b</td><td></td><td>A^2 is singular when $\det A^2 = 0$
$(16-2p)(1-2p)+50p = 0$
$16-34p+4p^2+50p = 0$
$4p^2+16p+16 = 0$
$4(p+2)^2 = 0$
$\boldsymbol{p = -2}$
OR
A^2 is singular when A is singular, [i.e. when $\det A = 0$]

$4+2p = 0$
$\boldsymbol{p = -2}$</td><td></td><td>•[2] Property stated or implied.[4]

•[3] Correct value of p.[5,1]

•[2] Explicitly states property. [not essential, but preferred]

•[3] Correct value of p[1]</td></tr>
<tr><td></td><td>c</td><td></td><td>$A'=\begin{pmatrix}4 & -2\\ p & 1\end{pmatrix}$
$\begin{pmatrix}x & -6\\ 1 & 3\end{pmatrix}=3\begin{pmatrix}4 & -2\\ p & 1\end{pmatrix}$
$\begin{pmatrix}x & -6\\ 1 & 3\end{pmatrix}=\begin{pmatrix}12 & -6\\ 3p & 3\end{pmatrix}$ $\quad x=12,\ p=\frac{1}{3}$</td><td></td><td>•[4] A transpose (A^T) correct. Does not have to be explicitly stated.

•[5] Values of p <u>and</u> x correct.[1,2]</td></tr>
<tr><td colspan="6">Notes:
3.1 For (a) and (c), statement of answers only: award full marks. For (b), $p = -2$ only, award •[3] only (1 out of 2)
3.2 Misinterpretation of A^T as inverse leading to $p = 0$ and $x = \frac{3}{4}$ OR to $p = -\frac{8}{3}$ and $x = -\frac{9}{4}$ OR to $p = 1$ and $x = \frac{1}{2}$ OR any other set of inconsistent equations: do not award •[4] or •[5] i.e. 0 out of 2.
3.3 Accept unsimplified answers.
3.4 Usually implied by next line.
3.5 For any equation based on answer to (a), correctly obtaining all possible solutions, including complex, •[2] •[3] both available. “No solutions”, “not possible” etc. •[3] not available, even if true.</td></tr>
</table>

Question			Expected Answer/s	Max Mark	Additional Guidance
4.			**The velocity, v, of a particle P at time t is given by** $v = e^{3t} + 2e^{t}$. **(a) Find the acceleration of P at time t.** **(b) Find the distance covered by P between $t = 0$ and $t = \ln 3$.**	 2 3	
	a		$a = \frac{dv}{dt}$ $= 3e^{3t} + 2e^{t}$		●¹ Evidence of knowing to differentiate.[2] ●² Correct completion.
	b		$s = \int_0^{\ln 3} v\,dt = \int_0^{\ln 3}\left(e^{3t} + 2e^{t}\right)dt$ $= \left[\frac{1}{3}e^{3t} + 2e^{t}\right]_0^{\ln 3}$ $= \left(\frac{1}{3}e^{3\ln 3} + 2e^{\ln 3}\right) - \left(\frac{1}{3} + 2\right)$ $= \frac{1}{3}e^{\ln 3^3} + 2e^{\ln 3} - \frac{7}{3}$ $= \frac{1}{3} \times 27 + 2 \times 3 - \frac{7}{3}$ $= \frac{38}{3}$ or $12\frac{2}{3}$ or equivalent		●³ Correctly set up integral. ●⁴ Integrating correctly. ●⁵ Evaluates correctly.[1,3]

Notes:

4.1 Accept rounded answers without working between ●⁴ and ●⁵ to 3s.f. or better.
Accept $12\cdot\dot{6}$, but not $12\cdot6$, 12 or 13.

4.2 Exceptionally, accept statement of formula as sufficient evidence for ●¹.

4.3 Evaluation of any incorrect function may be awarded ●⁵ if evaluation of at least one $e^{\ln q}$ involved.

4.4 Candidates may integrate v to obtain an expression for s and evaluate from there. Do not penalise the omission of "+ c".

Question			Expected Answer/s	Max Mark	Additional Guidance
5.			**Use the Euclidean algorithm to obtain the greatest common divisor of 1204 and 833, expressing it in the form 1204*a* + 833*b*, where *a* and *b* are integers.** $1204 = 1 \times 833 + 371$ $833 = 2 \times 371 + 91$ $371 = 4 \times 91 + 7$ $91 = 13 \times 7$ so gcd is 7 $7 = 371 - 4 \times 91$ $= 371 - 4(833 - 2 \times 371)$ $= 9 \times 371 - 4 \times 833$ $= 9(1204 - 1 \times 833) - 4 \times 833$ $= 9 \times 1204 - 13 \times 833$ ($a = 9$, $b = -13$)	**4**	•[1] Starting correctly. •[2] Obtains GCD. Accept (833, 1204) = 7 •[3] Equates GCD from •[2] and evidence of correct back substitution.[1,4] •[4] Correct form of final answer.[5]

Notes:

5.1 $7 = 371 - 4 \times 91$ not sufficient for •[3].
5.2 *a*, *b* do not need to be stated explicitly.
5.3 Accept a properly laid out grid approach.
5.4 Where candidate incorrectly starts "0=…" and correctly completes, •[4] available. Leads to $a = -119$ and $b = 172$ (or to $a = 119$ and $b = -172$).
5.5 For stating "a = 9, b = – 13" and nothing else, the question has not been answered, so 0/4.
5.6 Where $7 = 9 \times 1204 - 13 \times 833$, or arithmetically correct equivalent following from a wrong GCD with an inappropriate method or without supporting working, •[4] is available, but •[3] is not.

Question			Expected Answer/s	Max Mark	Additional Guidance
6.			**Integrate** $\dfrac{\sec^2 3x}{1+\tan 3x}$ **with respect to** x**.**	**4**	
			$\dfrac{f'(x)}{f(x)}$		•[1] Evidence knows correct form of integral.
			$=\dfrac{1}{3}....$		•[2] Coefficient correct.
			….ln….		•[3] Use of ln or $\log_e$.
			..... $\lvert 1+\tan 3x\rvert$		•[4] Completes, including use of |mod|[1]
			$=\dfrac{1}{3}\ln\lvert 1+\tan 3x\rvert + c$		
			OR		
			$u = 1 + \tan 3x$ **OR** $u = \tan 3x$		•[1] Correct substitution.
			$\dfrac{du}{dx} = 3\sec^2 3x$		•[2] Differentiates accurately.
			$\dfrac{1}{3}du = \sec^2 3x\, dx$		
			$\int \dfrac{\frac{1}{3}du}{u}$ **OR** $\int \dfrac{\frac{1}{3}du}{1+u} = \cdots$		•[3] Correct substitution of *du* and *f*(*u*) into integral.
			$=\dfrac{1}{3}\ln\lvert u\rvert + c$ **OR** $=\dfrac{1}{3}\ln\lvert 1+u\rvert + c$		
			$=\dfrac{1}{3}\ln\lvert 1+\tan 3x\rvert + c$		•[4] Integrates correctly *and* substitutes back.[1,2,3]

Notes:

6.1 Do not penalise omission of "+ c".

6.2 \|Modulus\|symbols necessary for •[4]

6.3 Accept $\frac{1}{3}\log\lvert 1+\tan 3x\rvert$ for full marks.

6.4 Accept answer without working for full marks.

6.5 Award $\ln\lvert 1+\tan 3x\rvert$ 3 marks out of 4.

Question			Expected Answer/s	Max Mark	Additional Guidance
7.			**Given that $z = 1 - \sqrt{3}\,i$, write down $\bar{z}$ and express $\bar{z}^2$ in polar form.** $\bar{z} = 1 + \sqrt{3}\,i$ $\bar{z} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ $\bar{z}^2 = \left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^2 = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ **OR** $\bar{z}^2 = \left(1 + \sqrt{3}\,i\right)^2 = 1 + 2\sqrt{3}\,i - 3 = -2 + 2\sqrt{3}\,i$ $\bar{z}^2 = -2 + 2\sqrt{3}\,i = r\left(\cos\theta + i\sin\theta\right)$ $r = 4, \theta = \frac{2\pi}{3}, \ \bar{z}^2 = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$	**4**	•[1] Correct statement of conjugate. •[2] One of r, θ correct.[1] •[3] Second correct and accurate substitution.[1] •[4] Processes to answer.[4,5] •[2] Obtains $\bar{z}^2$ in Cartesian form. •[3] One of r, θ correct. •[4] Second correct and accurate substitution.[4]

Notes:

7.1 Accept $2\text{cis}\frac{\pi}{3}$ for •[2] & •[3].

7.2 Accept angles expressed in degrees, i.e. 60°, 120°.

7.3 Where a candidate has applied de Moivre's theorem to $k(\cos\theta - i\sin\theta)$, do not penalise.

7.4 Correct polar form only. Answer in form $k(\cos\theta - i\sin\theta)$ loses •[4] unless correct form appears also.

7.5 Accept answers from $-\pi$ to 2π as being in polar form. For answers outside this range, do not award •[4].

7.6 Since it is possible to use the conjugate of z^2 to find $\bar{z}^2$ award •[2] for $z = 2\text{cis}\frac{5\pi}{3}$ or $2\text{cis}(-\frac{\pi}{3})$ and •[3] for $z^2 = 4\text{cis}\frac{4\pi}{3}$ or $4\text{cis}(-\frac{2\pi}{3})$, but only •[3] for $\bar{z}^2 = 4\text{cis}\frac{4\pi}{3}$ or $4\text{cis}(-\frac{2\pi}{3})$.

Question			Expected Answer/s	Max Mark	Additional Guidance
8.			**Use integration by parts to obtain** $\int x^2 \cos 3x\, dx$**.**	**5**	
			$\left[x^2.\frac{1}{3}\sin 3x\right] - \int \frac{2}{3}x \sin 3x\, dx$		•[1] Evidence of integration by parts.[1] •[2] Correct choice of *u, v'*. •[3] Accuracy of both expressions.
			$= \left[\frac{1}{3}x^2 \sin 3x\right] - \left[-\frac{2}{9}x \cos 3x - \int -\frac{2}{9}\cos 3x\, dx\right]$		•[4] Correct second application.
			$= \frac{1}{3}x^2 \sin 3x + \frac{2}{9}x \cos 3x - \frac{2}{27}\sin 3x + c$		•[5] Final integration and simplification.[4]

Notes:

8.1 Selection of parts wrong way round, leading to $\frac{1}{3}x^3\cos 3x - \int -\frac{1}{3}x^3.3.\sin 3x\, dx$ and no further, gains •[1] & •[3].

8.2 •[4] & •[5] available for follow through marks with a valid expression of equivalent difficulty.

8.3 Do not penalise omission of "+ *c*" in this case.

8.4 For a follow through mark to be awarded here, fractions and three (or more) trig functions are required.

Question			Expected Answer/s	Max Mark	Additional Guidance
9.			**Prove by induction that, for all positive integers n,** $\boldsymbol{\sum_{r=1}^{n}\left(4r^3+3r^2+r\right)=n\left(n+1\right)^3}$.	6	
			For $n = 1$ L.H.S $\sum_{r=1}^{n}\left(4r^3+3r^2+r\right) = 4 + 3 + 1 = 8$ R.H.S $n\left(n+1\right)^3 = 1 \times 2^3 = 8$ $\Rightarrow$ true for $n=1$		•[1] Evaluation of both sides independently to 8.[8]
			Assume true for $n = k$, $\sum_{r=1}^{k}\left(4r^3+3r^2+r\right)=k\left(k+1\right)^3$		•[2] Inductive hypothesis (must include "Assume true…" or equivalent phrase).[3,4]
			Consider $n = k + 1$, $\sum_{r=1}^{k+1}\left(4r^3+3r^2+r\right)$ $=\sum_{r=1}^{k}\left(4r^3+3r^2+r\right)+4\left(k+1\right)^3+3\left(k+1\right)^2+\left(k+1\right)$		•[3] Addition of $(k + 1)$th term.[5]
			$=k\left(k+1\right)^3+4\left(k+1\right)^3+3\left(k+1\right)^2+\left(k+1\right)$ $=\left(k+1\right)\left[k\left(k+1\right)^2+4\left(k+1\right)^2+3\left(k+1\right)+1\right]$		•[4] Use of inductive hypothesis *and* first step in factorisation process.[1,6]
			$=\left(k+1\right)\left[k\left(k^2+2k+1\right)+4\left(k^2+2k+1\right)+3\left(k+1\right)+1\right]$ $=\left(k+1\right)\left[k^3+2k^2+k+4k^2+8k+4+3k+3+1\right]$ $=\left(k+1\right)\left(k^3+6k^2+12k+8\right)$		•[5] Processing and simplifying to arrive at second factor.[1]
			$=\left(k+1\right)\left(k+2\right)^3$ $=\left(k+1\right)\left(\left(k+1\right)+1\right)^3$ Hence, if true for $n = k$, then true for $n = k + 1$, but since true for $n = 1$, then by induction true for all positive integers n.		•[6] Statement of result in terms of $(k + 1)$ and valid statement of conclusion.[1,7]

Notes:

9.1 Markers to take extra care to ensure that no steps are omitted in the algebra required for •[4], •[5] and •[6].

9.2 Alternative approach manipulating final form to "Aim: [or target] $k^4 + 7k^3 + 18k^2 + 20k + 8$" full marks are available, provided a valid conclusion is stated and working towards achieving "aim" is clear.

9.3 Acceptable phrases include: "If true for..."; "Suppose true for…"; "Assume true for…"; "Assume for $n = k$…". However, not acceptable would include: "Consider $n=k$"; "Assume $n = k$ …"; and "True for $n=k$".

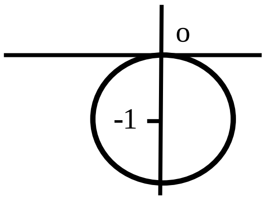
9.4 Correct statement of RHS in following lines where $k(k + 1)^3$ replaces $\sum_1^k(4r^3+3r^2+r)$, award •[2].

9.5 $(k + 1)$th term may appear later in working, but still achieves award of •[3].

9.6 "Aim: [or target] $k^4 + 7k^3 + 18k^2 + 20k + 8$" award •[4].

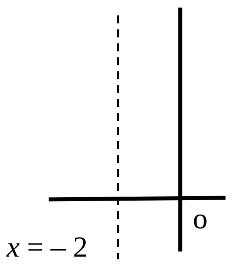
9.7 Acceptable form for •[6]: "If true for $n = k$, then true for $n = k+1$, but since true for $n = 1$, then true for all positive integers, n" or equivalent. Final line may be omitted if final line of algebra "$(k + 1)(k + 2)^3$" appears as aim/target.

9.8 "RHS = 8, LHS = 8" or "true for $n = 1$" are insufficient, on their own, for •[1].

Question			Expected Answer/s	Max Mark	Additional Guidance
10.			**Describe the loci in the complex plane given by:**		
			(a) $\lvert z+i \rvert = 1$	2	
			(b) $\lvert z-1 \rvert = \lvert z+5 \rvert$	3	
	a		Circle...		•[1] Observation that locus will be a circle.[8]
			...centre (0, – 1) [or – *i*], radius 1		•[2] Identification of centre[1] (in either form) and radius.[8]
			OR $z + i = x + iy + i = x + i(y + 1)$		
			$\lvert x+(y+1)i \rvert^2 = 1$		•[1] Correct expression for modulus in Cartesian form.[5]
			$x^2 + (y+1)^2 = 1$		
			Circle centre (0, – 1), radius 1		•[2] Statement that locus is a circle, centre[1] (in either form) and radius.
			OR 		•[1] Sketch of a circle •[2] Identification of centre and radius.[2,6]

Notes:

10a.1 (0, - *i*) not acceptable.
10a.2 For diagrammatic approach, radius must be stated or clearly just touch *x*-axis, with centre identified as (0, - 1).
10a.5 $\lvert x + (y+1)i \rvert = 1$ also acceptable for •[1].
10a.6 Where there is no point given and no point marked on the *y*-axis, do not award •[2].
10a.7 Where point on *y*-axis is identified as – *i*, •[2] may be awarded.
10a.8 Correct statement of centre and radius alone not sufficient for •[1]. Must explicitly state locus a circle or sketch.

Question			Expected Answer/s	Max Mark	Additional Guidance
10.			**(continued)**		
	b		Set of points equidistant from (1, 0) and (– 5, 0)		•³ Observation that equidistant from specified points.
			Straight line…		•⁴ Identifies form of locus.
			…$x = -2$		•⁵ Statement of equation.³
			OR $\lvert z-1\rvert^2 = \lvert z+5\rvert^2$		
			$\lvert(\boldsymbol{x-1})+\boldsymbol{iy}\rvert^{\mathbf{2}} = \lvert(\boldsymbol{x+5})+\boldsymbol{iy}\rvert^{\mathbf{2}}$ $(x-1)^2+y^2 = (x+5)^2+y^2$		•³ Collects real and imaginary parts *and* equates moduli.⁴,⁸
			$-2x+1 = 10x+25$ $-24 = 12x$ $x = -2$		•⁴ Accurately processes to reach equation.
			which is a **straight line**		•⁵ Explicitly states form of locus.³
			OR  		•³ Sketch of axes with any straight line drawn. •⁴ Vertical line to left of *y*-axis. •⁵ Explicitly states equation OR identifies the point (-2, 0) as being on the line.

Notes:

10b.3 Statement "line x = - 2" only, award full (3) marks. Statement "x = - 2" only loses •³ and •⁵, i.e. 1 out of 3.

10b.4 $\sqrt{(x-1)^2+y^2} = \sqrt{(x+5)^2+y^2}$ and no further, award •³, "Equates moduli", only.

10b.8 For this statement with squared "2" omitted from both sides, •³ may be awarded.

<table>
<tr><th colspan="3">Question</th><th>Expected Answer/s</th><th>Max Mark</th><th>Additional Guidance</th></tr>
<tr><td>11.</td><td></td><td></td><td>A curve has equation
$$\boldsymbol{x^2 + 4xy + y^2 + 11 = 0}$$
Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point (- 2, 3).

$2x + 4x\frac{dy}{dx} + 4y$ ...
$... + 2y\frac{dy}{dx} = 0 \qquad (\Delta)$
$2(-2) + 4(-2)\frac{dy}{dx} + 4(3) + 2(3)\frac{dy}{dx} = 0 \;\therefore\; \frac{dy}{dx} = \mathbf{4}$
OR $\frac{dy}{dx} = -\frac{2x+4y}{4x+2y} = -\frac{x+2y}{2x+y} \quad (\dagger) \quad \therefore \frac{dy}{dx} = \mathbf{4}$

Differentiating (Δ): $2 + 4x\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4\frac{dy}{dx}$ ...
$\ldots + 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0$

$\therefore 2 + 4(-2)\frac{d^2y}{dx^2} + 8(4) + 2(3)\frac{d^2y}{dx^2}\; 2(4)^2 = 0$
$\therefore \frac{d^2y}{dx^2} = \mathbf{33}$
OR Differentiating (†):
$$\frac{d^2y}{dx^2} = -\frac{(2x+y)\left(1+2\frac{dy}{dx}\right) - (x+2y)\left(2+\frac{dy}{dx}\right)}{(2x+y)^2}$$
$$\frac{d^2y}{dx^2} = -\frac{(2(-2)+3)(1+2(4)) - ((-2)+2(3))(2+4)}{(2(-2)+3)^2} = \mathbf{33}$$</td><td>6</td><td>•1 Differentiates x^2 and first product.
•2 Differentiates $y^2 + 11 = 0$ correctly.

•3 Evaluates $\frac{dy}{dx}$.[4]

•3 Evaluates $\frac{dy}{dx}$ after rearranging.

•4 Differentiates first three terms of (Δ) correctly, including a product.[3]
•5 Differentiates final product of (Δ) correctly.[3]

•6 Evaluates $\frac{d^2y}{dx^2}$.[1]
•4 Evidence of valid application of quotient (or product) rule.
•5 Differentiates correctly.

•6 Evaluates $\frac{d^2y}{dx^2}$.[1, 4]</td></tr>
<tr><td colspan="6">11.1 Where a carried error is incorporated, there must be at least one instance of evaluating $\frac{dy}{dx}$ to earn •6
11.2 Where the candidate has erroneously prefixed the question with “ $\frac{dy}{dx}$ =” and subsequently ignores = 0, leading to $\frac{dy}{dx} = \frac{8}{3}$ and $\frac{d^2y}{dx^2} = \frac{338}{27}$ award 5 out of 6, losing •2.
11.3 Where the candidate has simplified subsequent working with an error in first three marks, any three terms including a product could earn •4. A second product would make •5 also available.
11.4 Evaluation includes any simplification of differentiated expression in both parts. Therefore •3 and •6 awarded for correct evaluation of unsimplified derivative in each case.</td></tr>
</table>

Question			Expected Answer/s	Max Mark	Additional Guidance
12.			**Let n be a natural number.** **For each of the following statements, decide whether it is true or false.** **If true, give a proof, if false, give a counterexample.** **A** **If n is a multiple of 9 then so is n^2.** **B** **If n^2 is a multiple of 9 then so is n.**	**4**	
	A		Suppose $n = 9m$ for some natural number [positive integer], m. Then $n^2 = 81m^2 = 9(9m^2)$ Hence n^2 is a multiple of 9, so **A** is **true**.		$\bullet^1$ Generalisation, using *different* letter.[3, 6] $\bullet^2$ Correct multiplication *and* 9 extracted as a factor. $\bullet^3$ Conclusion of proof *and* state A true.[1]
	B		**False.** Accept any valid counterexample: $n = 3, 6, 12, 15, 21$ etc		$\bullet^4$ Valid counterexample *and* conclusion.[5]

Notes:

12.1 Final mark ($\bullet^3$) not available unless evidence of a proof attempted.

12.2 No credit given for numerical examples without generalisation.

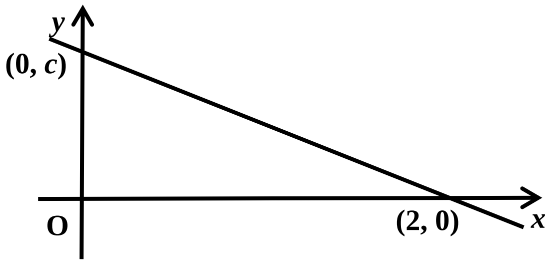
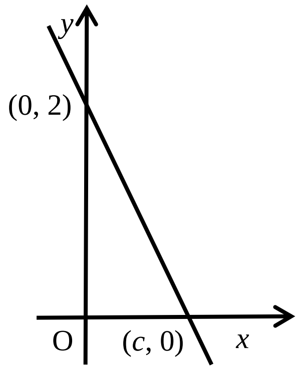
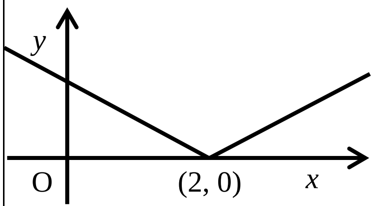
12.3 Do not penalise failure to specify that m is a natural number.

12.4 Any number of numerical examples on their own secures no marks.

12.5 Counterexamples must have n as a natural number (positive integer).

12.6 Starting $n^2 = (9n)^2$ i.e. using the same letter on both sides, leads inexorably to 0/3.

12.7 A 'word-based' proof is very unlikely to be awarded full marks for A as the criteria for $\bullet^2$ will be very difficult to achieve with words alone. A clear, logical answer of this type can access $\bullet^1$ and $\bullet^3$.

Question			Expected Answer/s	Max Mark	Additional Guidance
13.			**Part of the straight line graph of a function $f(x)$ is shown.**  **(a) Sketch the graph of $f^{-1}(x)$, showing points of intersection with the axes.** **(b) State the value of k for which $f(x) + k$ is an odd function.** **(c) Find the value of h for which $\lvert f(x+h) \rvert$ is an even function.**	 2 1 2	
	a		 		•[1] Straight line with negative gradient crossing the positive sections of the x- and y-axes. •[2] *Both* intersections correctly annotated.
	b		$y = f(x) - c$ is odd. $\therefore$ $\boldsymbol{k = -c}$		•[3] Correctly stated.
	c		 $y = \lvert f(x+2) \rvert$ is even $\therefore$ $\boldsymbol{h = 2}$		•[4] Sketch of $y = \lvert f(x) \rvert$ with point of reflection marked. •[5] Explicit statement of answer.

Notes:

13.1 Answer $h = 2$ only, no other working or diagram, award full marks [2 out of 2].
13.2 Where a candidate has clearly used their diagram from part (a) as the basis for (b) and (c), leading to $k = -2$ and $h = c$ (with working/further diagram) award •[4] •[5] and not •[3] (2 out of the three marks for (a) and (b)). Statement of above answers only, zero out of 3.
13.3 An accurate diagram of y = f(x + 2), on its own, gains no marks.

Question			Expected Answer/s	Max Mark	Additional Guidance
14.			**Solve the differential equation** $\boldsymbol{\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 4e^{3x}}$ **, given that $y = 1$ and** $\boldsymbol{\frac{dy}{dx} = -1}$ **when $x = 0$**	**11**	
			$m^2 - 6m + 9 = 0$ $(m-3)^2 = 0$ $m = 3$		$\bullet^1$ Correct auxiliary equation (or equivalent).[11]
			C.F. $y = Ae^{3x} + Bxe^{3x}$		$\bullet^2$ Correct solution of auxiliary equation ***and*** statement of complimentary function.
			P.I. Try $y = Cx^2e^{3x}$		$\bullet^3$ Correct form of particular integral.[1,7]
			$\frac{dy}{dx} = 2Cxe^{3x} + 3Cx^2e^{3x}$		$\bullet^4$ Correct first derivative of P.I.[2,3]
			$\frac{d^2y}{dx^2} = 2Ce^{3x} + 6Cxe^{3x} + 6Cxe^{3x} + 9Cx^2e^{3x}$		$\bullet^5$ Correct differentiation of first derivative.[4]
			$2Ce^{3x} + 6Cxe^{3x} + 6Cxe^{3x} + 9Cx^2e^{3x}$ $-6\left(2Cxe^{3x} + 3Cx^2e^{3x}\right) + 9Cx^2e^{3x} = 4e^{3x}$		$\bullet^6$ For correctly substituting expressions for both derivatives.
			$2Ce^{3x} = 4e^{3x} \Rightarrow C = 2$		$\bullet^7$ For correctly solving to obtain C.[5]
			G.S. $y = Ae^{3x} + Bxe^{3x} + 2x^2e^{3x}$		$\bullet^8$ Correct collation of above answers to obtain full General Solution.[6]
			$\frac{dy}{dx} = 3Ae^{3x} + Be^{3x} + 3Bxe^{3x} + 4xe^{3x} + 6x^2e^{3x}$		$\bullet^9$ Derivative of G.S.
			When $x = 0$, $y = 1$ $\quad A = 1$		$\bullet^{10}$ Use of i.c.s to find first constant correctly.
			$\frac{dy}{dx} = -1$ $\quad -1 = 3 + B \Rightarrow B = -4$		$\bullet^{11}$ Second constant.
			P.S. $\boldsymbol{y = e^{3x} - 4xe^{3x} + 2x^2e^{3x}}$		States solution.[6]

Question	Expected Answer/s	Max Mark	Additional Guidance

Question 14 Notes:

14.1 Accept $(Cx^2 + Dx + E)e^{3x}$ or equivalent for $\bullet^3$.

14.2 Accept correctly differentiated version of $(Cx^2 + Dx + E)e^{3x}$ or equivalent for $\bullet^4$.

14.3 Must be of equivalent difficulty if wrong P.I. used to obtain $\bullet^4$. i.e. contains at least one use of prod/quot rule.

14.4 Must be of equivalent difficulty if wrong 1st derivative used to obtain $\bullet^5$ i.e. contains at least one use of prod/quot rule.

14.5 And other constants, e.g. D and E = 0, where using alternative P.I.s as note 14.1.

14.6 Not needed if subsequent lines incorporate information, especially $\bullet^9$ or $\bullet^{11}$.

14.7 Incorrectly using $y = Cxe^{3x}$ for P.I. leading to A = 1 and B = - 4, $\bullet^4\,\bullet^6\,\bullet^9\,\bullet^{10}\,\bullet^{11}$ still available. i.e. max 7 (out of 11). To be awarded $\bullet^6$, a correct differentiation of the first derivative is required as well as correct substitution.

14.8 Incorrectly using $y = Ce^{3x}$ for P.I. leading to A = 1 and B = - 4, $\bullet^9\,\bullet^{10}\,\bullet^{11}$ still available. i.e. max 5 (out of 11).

14.9 Incorrectly using $y = Ae^{3x} + Be^{3x}$ and using $y = Ce^{3x}$ for P.I., $\bullet^9$ still available. i.e. max 2 (out of 11).

14.10 Incorrectly using $y = Ae^{3x} + Be^{3x}$ and using $y = Cxe^{3x}$ for P.I., $\bullet^4\,\bullet^9$ still available. i.e. max 3 (out of 11).

14.11 Do not penalise omission of "= 0" in first two lines.

Question			Expected Answer/s	Max Mark	Additional Guidance
15.			**(a) Find an equation of the plane π_1, through the points $A(0, -1, 3)$, $B(1, 0, 3)$ and $C(0, 0, 5)$.** **(b) π_2 is the plane through A with normal in the direction $-\mathbf{j} + \mathbf{k}$. Find an equation of the plane π_2.** **(c) Determine the acute angle between planes π_1 and π_2.**	4 2 3	
	a		$\overrightarrow{AB} = \begin{pmatrix}1\\1\\0\end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix}0\\1\\2\end{pmatrix}$ **OR** $\overrightarrow{BC} = \begin{pmatrix}-1\\0\\2\end{pmatrix}$		•1 Any two correct[1] vectors.[2]
			$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix}$ or equivalent		•2 Evidence of appropriate method.[3]
			$= 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$		•3 Obtains vector product (any form).
			$2x - 2y + z = 2 \times 0 - 2 \times -1 + 1 \times 3$		
			π_1 **: $2x - 2y + z = 5$** **OR** $r = \begin{pmatrix}0\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}0\\1\\2\end{pmatrix}$ or equivalent		•4 Obtains constant *and* states equation of plane.
	b		$0 \times 0 + (-1) \times (-1) + 1 \times 3 = 4$ π_2 **: $-y + z = 4$**		•5 Evidence of appropriate method.[4] •6 Processes to obtain equation of second plane.
	c		Normal vectors: $\boldsymbol{n}_1 = \begin{pmatrix}2\\-2\\1\end{pmatrix}$ and $\boldsymbol{n}_2 = \begin{pmatrix}0\\-1\\1\end{pmatrix}$, $\lvert\boldsymbol{n}_1\rvert = \sqrt{9} = 3, \lvert\boldsymbol{n}_2\rvert = \sqrt{2}$		•7 Obtains two correct lengths.
			cos (angle between normals) = $\dfrac{\boldsymbol{n}_1.\boldsymbol{n}_2}{\lvert\boldsymbol{n}_1\rvert\lvert\boldsymbol{n}_2\rvert} = \dfrac{2\times0-2\times-1+1\times1}{3\sqrt{2}} = \dfrac{3}{3\sqrt{2}} = \dfrac{1}{\sqrt{2}}$		•8 Evidence knows how to use formula.
			Angle $= 45^\circ$ acute angle between planes is $45^\circ \left(\text{or } \dfrac{\pi}{4}\right)$.		•9 Processes to statement of answer.[5]

P.T.O. for alternative method for question 15(c) and marking notes for all parts of question 15.

Question			Expected Answer/s	Max Mark	Additional Guidance
15.	**c**		**OR** $= 2i - 2j + k$, so $\lvert 2i - 2j + k \rvert = 3$ and $\lvert -j + k \rvert = \sqrt{2}$ $3 = \lvert n_1 \rvert . \lvert n_2 \rvert .\cos\theta = 3\sqrt{2}.\cos\theta$ $\cos\theta = \dfrac{1}{\sqrt{2}}$ so $\theta = \dfrac{\pi}{4}$ (or 45°)		•7 States vector *and* obtains moduli. •8 Evidence knows how to use formula. •9 Processes to statement of answer.

Notes:

15.1 i.e. non-parallel.
15.2 Although unconventional, accept vectors written horizontally.
15.3 Do not award •2 where co-ordinates of A/B/C used.
15.4 Award mark for obtaining value of constant = 4 (or follow-through).
15.5 Where candidate uses 90°- θ, only •7 and •8 available. So "θ = 45° so acute angle = 90°- 45°=45°" scores max of 2/3.

Question			Expected Answer/s	Max Mark	Additional Guidance
16.			**In an environment without enough resources to support a population greater than 1000, the population *P(t)* at time *t* is governed by Verhurst's law** $$\frac{dP}{dt} = P(1000 - P).$$ **Show that** $$\ln\frac{P}{1000-P} = 1000t + C \text{ for some constant } C.$$	**4**	
			Hence show that $$P(t) = \frac{1000K}{K + e^{-1000t}} \text{ for some constant } K.$$	**3**	
			Given that *P*(0) = 200, determine at what time *t*, *P*(*t*) = 900.	**3**	
			$\frac{dP}{dt} = P(1000 - P)$		
			So $\int \frac{dP}{P(1000-P)} = \int dt$		•1 Separates variables.[5]
			$\frac{1}{P(1000-P)} = \frac{A}{P} + \frac{B}{1000-P}$		•2 Appropriate form of partial fractions.
			$A = \frac{1}{1000}, B = \frac{1}{1000}$		•3 Obtains correct values of both *A* and *B*.
			$\frac{1}{1000}\int\left(\frac{1}{P} + \frac{1}{1000-P}\right) dP = \int dt$		
			$\ln P - \ln(1000 - P) = 1000t + c$		•4 Integrates correctly, including '+*c*'.[6]
			$\ln\frac{P}{1000-P} = 1000t + c$		
			$\frac{P}{1000-P} = Ke^{1000t}\left(where\ K = e^{c}\right)$		•5 Accurately converts to exponential form.[1]

Question			Expected Answer/s	Max Mark	Additional Guidance
16.			**(continued)** $P = 1000Ke^{1000t} - PKe^{1000t}$, $P + PKe^{1000t} = 1000Ke^{1000t}$, $P = \dfrac{1000Ke^{1000t}}{1 + Ke^{1000t}}$ $= \dfrac{1000K}{e^{-1000t} + K} \quad \left(\text{or } \dfrac{1000e^{c}}{e^{-1000t} + e^{c}}\right)$		•6 Multiplies out fractions and collects P terms. •7 Factorises and divides to obtain required form.[2]
			Since P(0) = 200, $200 = \dfrac{1000K}{1+K}$ $K = \dfrac{1}{4} \quad (\text{or } 0 \cdot 25)$ Require $900 = \dfrac{1000 \times 0 \cdot 25}{0 \cdot 25 + e^{-1000t}}$ $225 + 900e^{-1000t} = 250$ $e^{1000t} = 36$ $1000t = \ln 36$ $t = \dfrac{1}{1000}\ln 36$ [or 0·003584 (4sf)]		•8 Equates and process to obtain value of K.[3] •9 Inserts value of K *and* equates. •10 Solves to obtain value for t.[4]

Notes:

16.1 Explanation of new constant not required. Do not penalise bad form when changing constants. E.g., for RHS = $e^{1000t + c}$, award •5.
16.2 *Both* steps required as final result given in question.
16.3 Accept statement of value of K (consistent with previous working).
16.4 Accept approximation (2sf or better), ie 0·0036 or $3 \cdot 6 \times 10^{-3}$.
16.5 Do not penalise omission of integration symbols for •1.
16.6 Candidates putting ln$P$ + … lose •4.

Question			Expected Answer/s	Max Mark	Additional Guidance
17.			**Write down the sums to infinity of the geometric series** $\mathbf{1 + x + x^2 + x^3 + \dots}$ **and** $\mathbf{1 - x + x^2 - x^3 + \dots}$ **Valid for** $\lvert x \rvert < 1$. **Assuming that it is permitted to integrate an infinite series term by term, show that, for** $\lvert x \rvert < 1$, $\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\dots\right)$	**7**	
			Show how this series can be used to evaluate ln 2. **Hence determine the value of ln 2 correct to 3 decimal places.**	**3**	
			$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$		•[1] Correct statement of sum.
			$1 - x + x^2 - x^3 + \dots = \frac{1}{1+x}$		•[2] Correct statement of sum.
			Integrating the first of these gives: $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \dots = -\ln(1-x) + c$		•[3] Correct integration of both sides.[1]
			Putting $x = 0$ gives $c = 0$.		•[4] Correct evaluation of c.[3]
			Similarly, $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \dots = \ln(1+x)$		•[5] Correct integration of both sides.[1]
			Adding together gives: $2\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots\right) = \ln(1+x) - \ln(1-x)$		•[6] Evidence of appropriate method.
			$\left[= \ln\frac{1+x}{1-x}\right]$ as required.		•[7] Appropriate intermediate step.
			OR $2 + 2x^2 + 2x^4 + \cdots = \frac{1}{1+x} + \frac{1}{1-x}$		•[3] Adds series.
			$\therefore\ 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots$		•[4] Integrates LHS
			$= \ln(1+x) \dots$ $\dots - \ln(1-x) + c$		•[5] Integrates $\ln(1+x)$ •[6] Integrates $\ln(1-x)$

<table>
<tr><th colspan="3">Question</th><th>Expected Answer/s</th><th>Max Mark</th><th>Additional Guidance</th></tr>
<tr><td>17.</td><td></td><td></td><td>(continued)
Putting $x = 0$ gives $c = 0$.

$\left[= \ln\frac{1+x}{1-x}\right]$ as required.
OR</td><td></td><td>●7 Correct evaluation of c.[1,3]</td></tr>
<tr><td></td><td></td><td></td><td>$f(x) = ln\left(\frac{1+x}{1-x}\right)$ $f(0) = 0$

$f'(x) = 2(1 - x^2)^{-1}$ or equivalent $f'(0) = 2$

$f''(x) = 4x(1 - x^2)^{-2}$ $f''(0) = 0$

$f'''(x) = 16x^2(1 - x^2)^{-3} + 4(1 - x^2)^{-2}$ $f'''(0) = 4$

$f^{IV}(x) = 96x^3(1 - x^2)^{-4} + 48x(1 - x^2)^{-3}$ $f^{IV}(0) = 0$

$f^{V}(x) = 768x^4(1 - x^2)^{-5} + 576x^2(1 - x^2)^{-4} + 48(1 - x^2)^{-3}$ $f^{V}(0) = 48$

$\therefore f(x) = 0 + 2.1x + 0x^2 + \frac{4}{3!}x^3 + 0x^4 + \frac{48}{5!}x^5 + \cdots$

$= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots$

so $f(x) = \ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$ as required.</td><td></td><td>●3 Evidence of appropriate use of Maclaurin.[5,7]

●4 All five derivatives correct OR first two derivatives and first three evaluations correct.[5]
●5 All six evaluations correct OR final three derivatives correct and final three evaluations correct.[5]

●6 Correctly substitutes obtained values into Maclaurin.

●7 Simplification en route to required result.[8]</td></tr>
<tr><td></td><td></td><td></td><td>Now choose x such that $= \frac{1+x}{1-x} = 2,$

ie $1 + x = 2 - 2x$, so $x = \frac{1}{3}$

So $\ln 2 = 2\left(\frac{1}{3} + \frac{1}{81} + \frac{1}{1215} + \frac{1}{15309} + \ldots\right)$

$= 0{\cdot}693$ to 3 d.p.</td><td></td><td>●8 States appropriate equation.

●9 Correctly solves equation.[4]

●10 Obtains accurate approximation.[2,6]</td></tr>
</table>

Notes:

17.1 Do not penalise omission of '$+c$' for ●3 or ●5(or ●6 in alternative method). Mark for ●4 transferable: for the explicit evaluation of constant of integration on either (or both) of the integrals, award ●4 (or ●7 in alternative method).

17.2 To illustrate that series used (and not calculator), all 4 terms must appear for this mark to be awarded. For approximations, four or more decimal places in all four terms are required.

17.3 Any assumption that the two constants of integration are equal and can therefore be "cancelled out" loses ●4.

17.4 For an unsupported statement that $x = \frac{1}{3}$ award ●8 and ●9.

17.5 For attempts using Maclaurin's Theorem, only terms up to and including x^5 are required for ●3, ●4,and ●5.

17.6 Attempts utilising Maclaurin's Theorem do not need to continue to the term in x^7, as the wording of the question implies continuation of the established pattern. Hence errors in calculating the terms in x^6 and x^7 may be considered working subsequent to a correct answer.

17.7 Differentiation of either $\ln\left(\frac{1+x}{1-x}\right)$ or both of $\ln(1 + x)$ and $\ln(1 - x)$. Also accept differentiation of either ln expression and subsequent substitution of $(-x)$ for x to obtain the other (for ●3, ●4, and ●5), ●6 for combining and ●7 for simplifying. Substitution of $\left(\frac{1+x}{1-x}\right)$ for x is *not* accepted as it leads to an expansion which cannot be approximated in the same way.

17.8 Simplification to penultimate line or equivalent required for award of ●7 .

[END OF MARKING INSTRUCTIONS]