X100/701

NATIONAL QUALIFICATIONS 2010

FRIDAY, 21 MAY
1.00 PM – 4.00 PM

MATHEMATICS
ADVANCED HIGHER

Read carefully

1. Calculators may be used in this paper.
2. Candidates should answer **all** questions.
3. **Full credit will be given only where the solution contains appropriate working.**

Marks

Answer all the questions.

1. Differentiate the following functions.

(*a*) $f(x) = e^x \sin x^2$. **3**

(*b*) $g(x) = \dfrac{x^3}{(1 + \tan x)}$. **3**

2. The second and third terms of a geometric series are –6 and 3 respectively.

Explain why the series has a sum to infinity, and obtain this sum. **5**

3. (*a*) Use the substitution $t = x^4$ to obtain $\int \dfrac{x^3}{1 + x^8}\,dx$. **3**

(*b*) Integrate $x^2 \ln x$ with respect to x. **4**

4. Obtain the 2 × 2 matrix M associated with an enlargement, scale factor 2, followed by a clockwise rotation of 60° about the origin. **4**

5. Show that

$$\binom{n+1}{3} - \binom{n}{3} = \binom{n}{2}$$

where the integer n is greater than or equal to 3. **4**

6. Given $\mathbf{u} = -2\mathbf{i} + 5\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{w} = -\mathbf{i} + \mathbf{j} + 4\mathbf{k}$.

Calculate $\mathbf{u}.\,(\mathbf{v} \times \mathbf{w})$. **4**

Marks

7. Evaluate

$$\int_1^2 \frac{3x+5}{(x+1)(x+2)(x+3)}\,dx$$

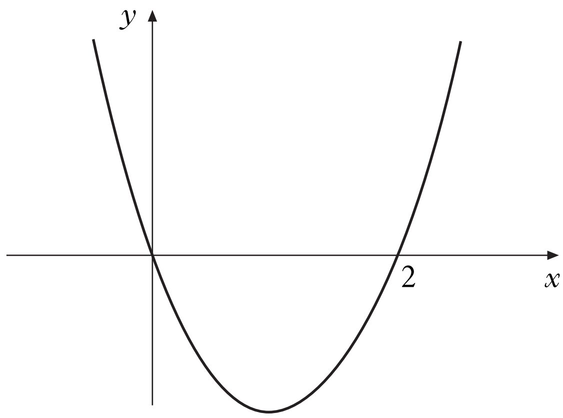
expressing your answer in the form $\ln\frac{a}{b}$, where a and b are integers. **6**

8. (*a*) Prove that the product of two odd integers is odd. **2**

(*b*) Let p be an odd integer. Use the result of (*a*) to prove by induction that p^n is odd for all positive integers n. **4**

9. Obtain the first three non-zero terms in the Maclaurin expansion of $(1 + \sin^2 x)$. **4**

10. The diagram below shows part of the graph of a function $f(x)$. State whether $f(x)$ is odd, even or neither. Fully justify your answer.



3

11. Obtain the general solution of the equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0.$$ **4**

Hence obtain the solution for which $y = 3$ when $x = 0$ and $y = e^{-\pi}$ when $x = \frac{\pi}{2}$. **3**

Marks

12. Prove by contradiction that if x is an irrational number, then $2 + x$ is irrational. **4**

13. Given $y = t^3 - \frac{5}{2}t^2$ and $x = \sqrt{t}$ for $t > 0$, use parametric differentiation to express $\frac{dy}{dx}$ in terms of t in simplified form. **4**

Show that $\frac{d^2y}{dx^2} = at^2 + bt$, determining the values of the constants a and b. **3**

Obtain an equation for the tangent to the curve which passes through the point of inflexion. **3**

14. Use Gaussian elimination to show that the set of equations

$$\begin{aligned} x - y + z &= 1 \\ x + y + 2z &= 0 \\ 2x - y + az &= 2 \end{aligned}$$

has a unique solution when $a \neq 2{\cdot}5$. **5**

Explain what happens when $a = 2{\cdot}5$. **1**

Obtain the solution when $a = 3$. **1**

Given $A = \begin{pmatrix} 5 & 2 & -3 \\ 1 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, calculate AB. **1**

Hence, or otherwise, state the relationship between A and the matrix

$$C = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}.$$ **2**

Marks

15. A new board game has been invented and the symmetrical design on the board is made from four identical "petal" shapes. One of these petals is the region enclosed between the curves $y = x^2$ and $y^2 = 8x$ as shown shaded in diagram 1 below.

Calculate the area of the complete design, as shown in diagram 2. **5**

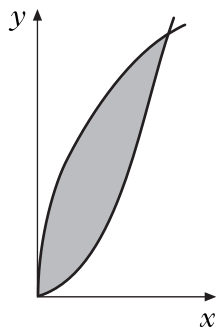


Diagram 1

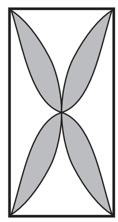


Diagram 2

The counter used in the game is formed by rotating the shaded area shown in diagram 1 above, through 360° about the y-axis. Find the volume of plastic required to make one counter. **5**

16. Given $z = r(\cos\theta + i\sin\theta)$, use de Moivre's theorem to express z^3 in polar form. **1**

Hence obtain $\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^3$ in the form $a + ib$. **2**

Hence, or otherwise, obtain the roots of the equation $z^3 = 8$ in Cartesian form. **4**

Denoting the roots of $z^3 = 8$ by z_1, z_2, z_3:

(a) state the value $z_1 + z_2 + z_3$;

(b) obtain the value of $z_1^6 + z_2^6 + z_3^6$. **3**

[*END OF QUESTION PAPER*]

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